ELEMENTARY GEOMETRY

SECTION I

PARTS I & II

BARRELL

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ELEMENTARY GEOMETRY

ELEMENTARY GEOMETRY

By **FRANK. R. BARRELL**, M.A., B.Sc., late Scholar of Pembroke College, Cambridge, Professor of Mathematics, University College, Bristol.

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ELEMENTARY GEOMETRY

BY

FRANK. R. BARRELL, M.A., B.Sc.

LATE SCHOLAR OF PEMBROKE COLLEGE, CAMBRIDGE
FORMERLY INSTRUCTOR IN NATURAL SCIENCE IN H.M.S. "BRITANNIA"
PROFESSOR OF MATHEMATICS, UNIVERSITY COLLEGE, BRISTOL

SECTION I. PARTS I. AND II.

BEING THE SUBJECT-MATTER OF EUCLID, BOOK I.
BOOK III. 1–34, AND BOOK IV. 4–9

NEW EDITION

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LIST OF SYMBOLS USED

+ plus, or "together with."
− minus, or "diminished by."
= "is equal to," or "are equal to."
≡ "is equal in all respects to."
∠ angle.
△ triangle.
▱ parallelogram.
⊙ circle.
∥ parallel.
∴ therefore.
∵ because, or since.
⊥ perpendicular.
> is greater than.
< is less than.

NOTE 1.—In the last two symbols the point, which is the smaller end of the symbol, is next to the smaller of the quantities compared.

NOTE 2.—In the figures right angles are distinguished by the use of a double arc "in the corner."

GREEK LETTERS.

α,	pronounced	Alpha,	corresponds to	English	*a*.
β,	„	Beta,	„	„	*b*.
γ,	„	Gamma,	„	„	*g*.
δ,	„	Delta,	„	„	*d*.
θ,	„	Theta,	„	„	*th*.
ϕ,	„	Phi,	„	„	*ph*.
λ,	„	Lambda,	„	„	*l*.
μ,	„	Mu,	„	„	*m*.

Gamma is the third letter in the Greek alphabet, in which there is no *c*, hence α, β, γ, δ correspond to *a*, *b*, *c*, *d*.

INTRODUCED IN SECTION II.

SYMBOL. ⌙ "the difference between."

GREEK LETTERS—

π,	pronounced	Pi,	corresponds to	English	*p*.
ω,	„	Omega,	„	„	*ō*.

PREFACE

At the present time of upheaval in the methods of mathematical teaching, no reason need be offered for the production of a School Geometry.

The essential features which should, I think, characterize Geometrical Teaching are—

(1) Careful and accurate construction of all figures.

(2) Rejection of the unnecessary verbiage which has cumbered recent teaching of Euclid.

(3) Recognition of the fact that difficult points must be faced, and not shirked or slurred over.

Holding these views, I have not hesitated to discuss all fundamental ideas fully; not only young boys, but even children, try to puzzle out these questions, in the days before their desire for knowledge is choked off by cut-and-dried definitions framed in unintelligible language.

In the selection and arranging of propositions, I have been guided mainly by the New Syllabus for the Cambridge Local Examinations, and by the Report of the Committee of the Mathematical Association. Happily, my own views accord so closely with those of these two authorities, that this guidance has been free from any taint of coercion.

Propositions definitely omitted by the Cambridge Syllabus are marked with a single *, and those also omitted by the Mathematical Association, with double **.

Some of my friends object to my "third meaning of angle;" nevertheless I have retained it, for I believe that this concept is certainly implied in many phrases of the moribund "Euclids;" and, indeed, in spite of my *preference* for the rotation-definition, I suspect that the latter is truly only a definition of "angle-measurement," not of "angle."

Some half-hearted reformers regard the rotation *proof* of I. 32 as illustration only, yet they accept a rotation definition of angles. This position is untenable, for even if the old *proof* may not require rotation about different points, yet the mere *enunciation* of the proposition implies the power to add angles with different vertices, *i.e.* to add rotations about different points.

The constructions of Chapter V. should be taken quite early —possibly even before Chapter I. is read—the proofs being held over until the earlier chapters have been mastered.

A short chapter on Loci, with hints towards solving problems, is placed at the end of Part II. of Section I.; much of it may be read earlier if desired.

Single letters have been used wherever possible to indicate areas, lines, and angles. To guard against any difficulty in using the eight Greek letters selected, a list of these is given with other symbols: also each is given twice early in the book, at the head of pages where it appears.

It is of supreme importance that approximation methods should be used in calculating areas and in working square roots. These methods are illustrated by worked examples; for explanation of them reference must be made to a good Arithmetic.

My thanks are due to Rev. J. C. P. Aldous, M.A., late Chief Instructor in H.M.S. *Britannia;* to Mr. J. F. McKean, M.A., of the R.N. Engineering College; to my colleague

Mr. E. L. Watkin, M.A.; and to Miss Amy Cutbush, of the Royal Masonic School for Girls, for very careful reading of proofs and for other help; and I am also indebted to Mr. W. D. Eggar, M.A., of Eton College, for several valuable suggestions.

Those exercises which are not original are taken from the examination papers of various public bodies, mainly from the Cambridge Local Examinations.

In the answers to the practical questions, measures of lines and areas are given to three significant figures, and measures of angles to the nearest tenth of a degree, this being the ideal accuracy which could be reached in schoolwork.

It should be noted that in all figures some distinguishing mark is given to each pair of lines or of angles which are *known* to be equal; and also that these marks are given only when such equality is stated in the enunciation of the proposition, or is an immediate consequence of it.

In blackboard teaching, it is, of course, possible for the instructor to reduce the number of letters and symbols used in the figures, by replacing dissimilar symbols by similar ones at each stage of the proof at which some fresh equality has been established. But I believe that I have used as few symbols as is consistent with accuracy for written or printed work.

I shall be grateful for any corrections or criticisms sent to me.

FRANK. R. BARRELL.

UNIVERSITY COLLEGE, BRISTOL,
March, 1903.

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2. If from a point on a line there be drawn two other straight lines on opposite sides of it, making the two adjacent angles together equal to two right angles, then these two lines are parts of the same straight line. [*I. 14.*]

3. If two straight lines cut one another, the vertically opposite angles are equal. [*I. 15.*]

4. If each side of a convex polygon be produced in the same way, the sum of the exterior angles thus formed is equal to four right angles.

5. The three angles of a triangle are together equal to two right angles. [*I. 32.*]

6. Two straight lines which are parallel when tested by one transverse line are parallel when tested—

(1) by any transverse line which meets the first;

(2) by any transverse line which is not known to meet the first.

7. Two parallel straight lines do not meet when produced.

8. Lines which are parallel to the same line are parallel to each other. [*I. 30.*]

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9. If two sides and the angle contained by them in one triangle be equal to two sides and the angle contained by them in another triangle, then the triangles are equal in all respects. [*I. 4.*]

10. In an isosceles triangle, the angles opposite to the equal sides are equal. [*I. 5.*]

11. If two angles of a triangle be equal, the triangle is isosceles. [*I. 6.*]

12. If the three sides of one triangle be equal to the three sides of another triangle, then the triangles are equal in all respects. [*I. 8.*]

13. If two angles and a side of one triangle be equal to two angles and the corresponding side of another triangle, the triangles are equal in all respects. [*I. 26.*]

14. If two right-angled triangles have their hypotenuses equal, and also a side of one equal to a side of the other, the triangles are equal in all respects.

15. If two sides of one triangle be equal to two sides of another triangle, and if the angles opposite to one pair of equal sides be equal, then the angles opposite to the other pair are *either* (a) equal *or* (b) supplementary.

16. If one side of a triangle be greater than another, the angle opposite to the greater side is greater than the angle opposite to the smaller side. [*I. 18.*]

17. If one angle of a triangle be greater than another, the side opposite to the greater angle is greater than the side opposite to the smaller angle. [*I. 19.*]

18. Any two sides of a triangle are together greater than the third side. [*I. 20.*]

19. If two sides of one triangle be equal to two sides of another triangle, but the contained angles be not equal, then the triangle which has the greater contained angle has also the greater base. [*I. 24.*]

20. If two sides of one triangle be equal to two sides of another triangle, but the bases be not equal, then the triangle which has the greater base has also the greater angle opposite to it. [*I. 25.*]

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22. In any parallelogram (1) opposite angles are equal, (2) opposite sides are equal, (3) the diagonals bisect the parallelogram, (4) they bisect each other. [*I.* 34.]

23. If the opposite sides of a quadrilateral be equal, the quadrilateral is a parallelogram.

24. If two parallelograms have the same base and extend to the same straight line parallel to it, the parallelograms are equal in area. [*I.* 35.]

25. If two rectangles have equal bases and equal heights, they are equal in area.

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3. Bisect a given angle. [*I.* 9.]

4. Bisect a given straight line. [*I.* 10.]

5. Draw a straight line at right angles to a given line from a given point in it. [*I.* 11.]

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7. Draw through a given point a straight line parallel to a given straight line. [*I.* 31.]

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11. Construct a triangle equal to a given triangle and having an angle equal to a given angle.

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30. If two triangles of equal area have the same base and lie on opposite sides of it, the line joining their vertices is bisected by the common base.

31. If a line be drawn parallel to a side of a triangle, the part of it cut off by the other sides is bisected by the corresponding median.

32. If two vertices B, C of a triangle ABC be joined to any point P on the median through the third vertex A, the triangles ABP, ACP thus formed are equal in area.

33. The medians of a triangle are concurrent.

34. The straight lines drawn at right angles to the sides of a triangle through their middle points are concurrent.

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6. If one of the interior angles made by two straight lines with any transverse line be equal to the corresponding exterior angle, the former lines are parallel.

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(2) *Converse.*—If a diameter be at right angles to a chord which is not a diameter, the diameter bisects the chord. [*III. 3.*]

38. (1) Equal chords of a circle are equally distant from the centre.
(2) *Converse.*—Chords which are equidistant from the centre are equal. [*III. 14.*]

39. (1) If one of two chords of a circle be nearer to the centre, it is the greater chord.
(2) *Converse.*—If one of two chords be greater than the other, it is nearer to the centre.
(3) A diameter is greater than any chord which is not a diameter. [*III. 15.*]

40. In a circle, the angle at the centre which stands on any arc is double any angle at the circumference standing on the same arc. [*III. 20.*]

41. All angles in the same segment of a circle are equal. [*III. 21.*]

42. The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to two right angles. [*III. 22.*]

43. If the straight line joining two points subtend equal angles at two other points on the same side of it, then the four points are concyclic.

44. If the sum of a pair of opposite angles of a quadrilateral be equal to two right angles, the quadrilateral is cyclic.

45. (1) The angle in a semicircle is a right angle.
(2) The angle in a segment greater than a semicircle is less than a right angle.
(3) The angle in a segment less than a semicircle is greater than a right angle. [*III. 31.*]

46. In equal circles, (1) if two arcs subtend equal angles at the centre, then the arcs are equal;
Converse, (2) If the arcs be equal, then the angles are equal. [*III. 26, 27, 28, 29.*]

47. If from a point—not the centre—with a circle, straight lines be drawn to the circumference—
(1) the greatest of these lines passes through the centre:
(2) the least is opposite to the greatest;
(3) of any two others, the one which subtends the greater angle at the centre is the greater. [*III. 7.*]

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48. If st. lines be drawn to a circle from a point outside it—
(1) the greatest is the one which passes through the centre ;
(2) the shortest is the one which when produced passes through the centre ;
(3) of any two others, the one which subtends the greater angle at the centre is the greater. [*III. 8.*]

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16. Describe a circle about a given triangle. [*IV. 5.*]

17. Draw the tangent at a given point on a given circle.

18. Draw a tangent to a circle from a point outside it. [*III. 17.*]

19. Draw a direct common tangent to two given circles.

20. Draw a transverse common tangent to two given circles.

21. Inscribe a circle in a given square. [*IV. 8.*]

22. Describe a circle about a given square. [*IV. 9.*]

23. Inscribe a square in a given circle. [*IV. 6.*]

24. Describe a square about a circle. [*IV. 7.*]

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GEOMETRY

CHAPTER I.

INTRODUCTION.

THE universe in which we exist is filled with a countless variety of things of widely different character—vast bodies such as the sun, moon, and stars, large bodies such as mountains, smaller bodies such as animals, trees, or chairs, and again minute bodies visible only under a microscope. All these things, however widely they differ from one another, are alike in one respect, namely, they must "be" somewhere; if they exist, there must be a place for them to exist in, and they fill or occupy that place: in other words, bodies occupy space. Also all bodies are alike in that they have shape; the shape may be unchanging, as the shape of a statue; it may change slowly, as does the shape of a growing plant; or it may change rapidly, as does the shape of water in a stormy sea; still, at any instant, every particular body has its own particular shape, such as an instantaneous photograph would show.

The object of geometry is to study the shapes and the sizes of things—not of all things, however, for the shapes of most living things and of many lifeless things are too complicated and irregular for our study, and we shall only consider the more simple and comparatively regular shapes.

The best-known shapes of solid bodies are shown below. Notice the number of corners, edges, and faces which each

shape has. Also notice whether the edges are straight or curved, whether the faces are flat or curved, and which faces have square corners.

Thus the cylinder has 2 edges both curved, but has no corners.

The common pyramid has 5 corners, 8 edges all straight, and 5 faces all flat.

Each of the solids should be carefully studied to find out what shapes can be formed by slicing it completely through, or by cutting off corners, etc. Also notice whether it is possible to draw straight lines on the various surfaces, and if so, notice on which faces they can be drawn; *e.g.* on the sphere no straight lines can be drawn.

Thus a picture is given of the sphere with a piece sliced off the top; the flat face made by the cut is a circle.

If a corner of the cube be sliced off as shown shaded in the picture, the new face formed is a triangle.

A full discussion of these solids will be given in Section III. of this book; here we only mention a few points which cannot be shown well in the pictures.

The **cube** has all its edges of the same length.

The **common cylinder** has circles for its flat base and flat top.

The **common cone** has a circle for its flat base.

The **tetrahedron** is also called a **triangular pyramid.**

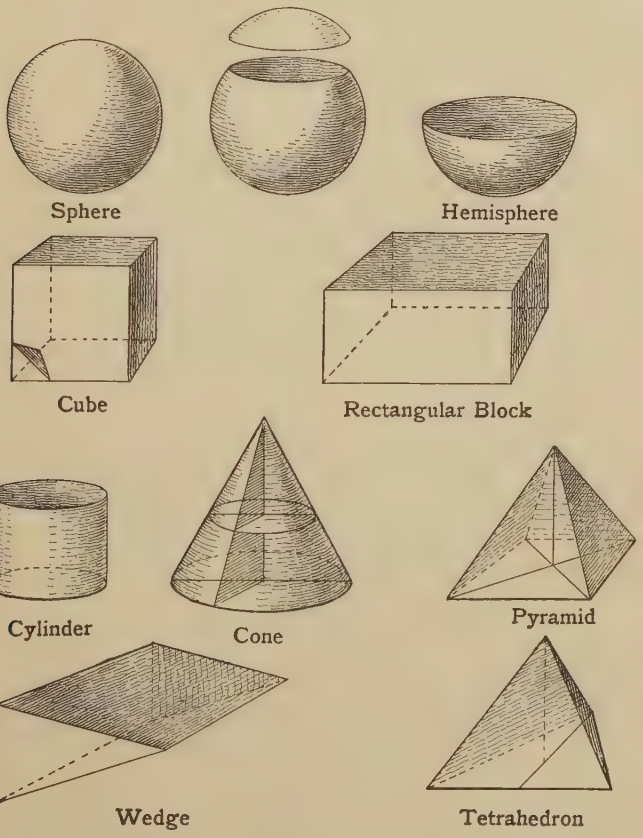
The common **pyramid** has a square base, and is often called a **square pyramid.**

The **wedge** has triangles for its flat ends, and is also called a **triangular prism.**

Use of Geometry.

Why should we study geometry? What good will it do us? These two questions are so frequently asked, not only by beginners, but by many who have "learnt Euclid" for years, that it may be well to answer them at the outset. Geometry has two distinct sides and two distinct uses—the practical and

the theoretical. On the practical side it teaches how to construct—*i.e.* to build up or draw—accurate pictures or figures of circles and squares, spheres, cylinders, and the like, and



teaches us how to draw them of any required size and in any required positions with regard to each other.

The utility of this to engineers, architects, and all who have to make drawings of machines and buildings is obvious; without accuracy, good workmanship cannot exist. But accurate

measured drawing is almost as useful to everybody, on account of the training it gives to the eye and the hand; the man who, when he wants a circle or a square, is in the habit of drawing a real circle or a real square and not a deformed caricature of them, has gained the power of appreciating good work and of detecting the slovenliness of bad work. This power adds much to the pleasure and interest of life, just as accuracy of eye and accuracy of hand give pleasure when concerned with tennis, cricket, or golf.

The theoretical side of geometry teaches us to discover facts concerning the figures which the practical side teaches us to construct; it teaches us how we may often decide which lines or angles or areas in a figure must be equal, or which must be the greater, without our having actually measured them; and it teaches us to predict before drawing such lines or angles which of them will be equal or greater, and so on.

But the more important benefit of theoretical geometry is, that it not only teaches us to assert that certain facts are facts, but it teaches us to fight for our assertions—with tongues and brains, not with fists—that is, it teaches us to argue or reason correctly. The art of reasoning correctly is called "logic." Molière makes a character in one of his plays express surprise on discovering that for fifty years he had been "talking prose" without knowing it. After studying geometry, any boy or girl should be able to "talk logic" for fifty years—whether consciously or unconsciously will not much matter.

For many years the books written by the Greek mathematician Euclid have been used for teaching geometry. In them various propositions are made. They consist of (1) proposals to **do** some required thing, *i.e.* of **problems** in practical geometry; (2) of statements that something is true —*i.e.* of **theorems** in theoretical geometry.

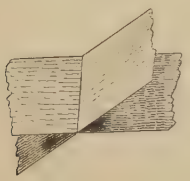
In **problems**, when we believe we have "done" or "constructed" what is required, we must **prove** that we have done so. In **theorems** we have to prove that the statements made are really true.

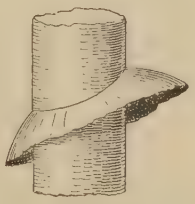
Many of the propositions in Euclid have elegant constructions and proofs; they are so well known to mathematicians by their place in Euclid's Books, that when giving the same propositions we shall quote their book and number in Euclid (thus, I. 47, meaning Euclid, Book I. Prop. 47). But several of his propositions are of little practical or theoretical use, and in a few the proofs are unsatisfactory; these we shall omit.

Surfaces, Lines, Points.

Consider any body such as a ball or a box: it occupies a part of space; the shape of this space is a **solid figure.**

There is a boundary separating the space "where the body is" from the space "where it is not:" this boundary has length and breadth, but no thickness; we call it a **surface.** If two surfaces intersect (*i.e.* meet or cross one another), their place of intersection is called a **line.** One of the figures shows the





line of intersection of two flat surfaces, the other shows the curved line of intersection of a tubular surface and a shell. The figures help us to see that lines have length, but no breadth or thickness. If two lines intersect one another, the place where they cross is called a **point.** A point has neither length, breadth, nor thickness.

Hence—

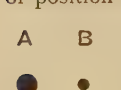
A **surface** is the **boundary of a portion of space.**
„ **line** „ **intersection of two surfaces.**
„ **point** „ „ „ **lines.**

Here we started with the idea of a solid, and obtained from it our ideas of surface, line, and point. The reverse

process is instructive, but as it is perhaps more difficult, the following pages may be omitted in a first reading of the subject.

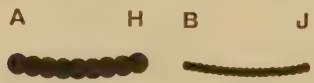
Points, Lines, Surfaces.

Consider this set of dots. Each one occupies a certain place or position on the paper. Each one has a certain size, but each dot is smaller than the preceding one. Now, below **E** imagine another dot, smaller than any of **A, B, C, D,** smaller than anything you can think of, even the smallest spot visible under a microscope. Such a dot, smaller than any limit you can fix, is infinitely (*i.e.* unboundedly) small. We call it a **point.** It has **no size**, but it has a **position.**



When we speak of the "point" of a needle or of a pencil, we are generally thinking of a small piece of it near the sharp end. This is not a geometrical point, for it is of appreciable size; nor is the mark made by a pencil a true point. In a book the position of a point must be indicated by something visible, hence we use either a visible dot, or a pair of short fine lines cutting each other at the point; we name points by means of letters placed near them.

Next consider a row of dots so close together that they overlap each other. We have a longish narrow mark **AH**. If the dots be



made smaller (as **BJ**), we must put more of them in the same length before they overlap, and they give us a narrower mark. Now below **CK** imagine another row of dots, each infinitely small; *i.e.* imagine a row of points. An infinitely large number of them must be taken before they overlap; but if you *can* imagine such a row of overlapping points, you are imagining a mark which has length, but no width. Such an infinitely narrow mark we call a **line.**

Again, consider an inky dot **A**, and push it along the paper to the position **H**. As it moves along it generates (*i.e.* makes) a mark **AH** similar to the previous mark **AH**, but more smooth along its sides. Push a smaller dot from **B** to **J**; we have a narrower mark **BJ.** Lastly, imagine a point below **C**; imagine it

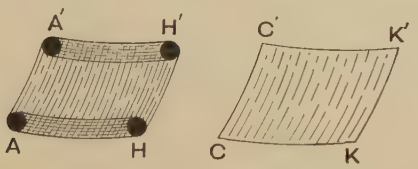
pushed along to a position below **K**. We cannot see any mark made by it, but we can imagine its infinitely narrow track. This

track has a **length** and a **position**, but it has **no width**, and, as before, we call it a **line.**

Thus there are two distinct ways of thinking of a geometrical line : (1) as a row of infinitely small points—an infinite number of them ; (2) as the track of a single point which has travelled from one position to another.

A geometrical line differs from, say, a fishing-line, in that it is necessarily invisible ; also the fishing-line, if cut across, shows a more or less round section, whereas the section of a true line is only a single point. To indicate the position of a line in figures we make marks which necessarily have some width ; but the finer we make our so-called lines the more correct our representation will be.

Next take an inky mark **AH** and push it broadside to the position **A′H′**. As it moves it generates a mark having both

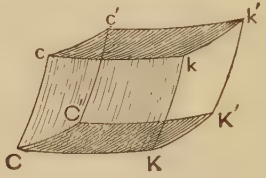


length and breadth, but no thickness (except for the thickness of the layer of ink). Or, again, imagine a line near **CK** pushed broadside along the paper to a position near **C′K′** ; the track which it has swept over has **length** and **breadth**, but **not thickness.** In each case we call the track a **surface.**

A surface may also be considered as built up of an infinite number of lines ; or, again, as a "doubly" infinite number of points, for each of the lines has itself an infinite number of points.

The surfaces shown above are flat, because the paper is flat. Had the paper been not-flat, the surfaces would be not-flat.

Finally, lift the surface **CKK'C'** to the position **ckk'c'**. The portion of space which is its "track" is a **solid figure.**

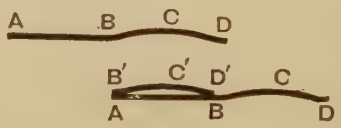


A solid may be considered as built up of an infinitely great number of surfaces; or, again, as a "doubly" infinite number of lines, for each surface is an infinite number of lines; or, lastly, as a "trebly" infinite number of points.

Straight Lines.

Lines are either straight or curved. Everybody *knows* what a straight line is; but how can we *define* a straight line? That is, how can we state exactly the difference between a line which is straight and one which is not? or what test can we give by which to recognize a straight line?

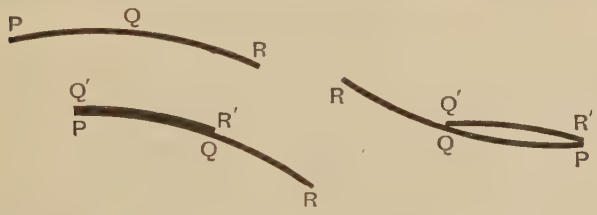
For example, **ABCD** is not straight. Why not? Because there is a "hump" at **C**. But what is a "hump"? Merely a



place where the line is not straight. So far, nothing has been gained; we have only decided "that the line isn't straight because it isn't straight."

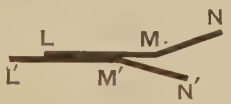
Now, however, make a copy **B'C'D'** of the part **BCD**—say, by tracing, or by bending a piece of fine wire to its shape; shift **ABCD**, and try to fit **AB** along **B'C'D'**. The gap between the lines proves that there is a hump in one of them. If the line is straight, there is no gap when it is placed along a copy

of any part of itself. Again, PQR is not straight; for although it fits the copy Q′R′ without a gap when moved along it as



ABCD was moved, yet a gap appears when it is placed along Q′R′ in the reversed position RQP.

Also LMN is not straight, for when turned over and placed



"back to back" along its copy L′M′N′, the parts MN, M′N′ open out or diverge from each other.

Definition.—A line is **straight** if, when a copy of any part of it is placed along the line in all possible ways, there is **no space** between it and the copy.

NOTE.—The definition in Euclid, "A straight line is one which lies evenly between its extremities," is worthless. A definition should either tell us some means by which to distinguish some thing from all other things, or it should explain the meaning of some new or half-understood word by the help of other words which are better understood. Now, everybody understands the word "straight" as well as, if not a little better than, the word "even;" and to define "straight" as "even" is to commit the error known in Logic as defining "ignotum per ignotius."

Sometimes the definition is given, "A straight line is the shortest line that can be drawn between its two extremities." This is true as a fact, but it is not good as a definition; it puts the cart before the horse. It tells us how to detect the shortest line, namely, by drawing a straight line; but it is not a true test of straightness, for we cannot find out which is the shortest route from a point A to a point B unless we measure every one of the myriads of possible routes. Still, the *fact* that "the shortest line is

straight" is often useful; for example, gardeners use a stretched string to guide them in digging a straight trench, and in Physics a stretched silk

fibre or a "spider line" is often used to indicate the position of a straight line.

When we judge of the straightness of a line, say the edge of a box, by looking along it end on, to see if it "looks straight," we rely on our conviction that light travels straight from the far end of the line to our eye, and we observe whether there is any gap between the actual line and the route which our eye tells us the light follows.

Equality of Lines.

We call lines equal, if they are of the same length. How shall we test this? The test for curved lines must be deferred, but for straight lines it is given in the following definition.

Definition.—Two straight lines are **said to be equal** if one of them (or a copy of it), when moved up to the other, can be **made to coincide** with it, *i.e.* if they can be made to fit exactly, so that both lines begin at the same point and both end at the same point.

Definition.—If points P, Q be taken between the ends of

a line AB, then AP, PQ, etc., are parts of AB. The **whole line is said to be greater than** any one of its **parts.**

Definition.—A **line** AB is **said to be greater than a line** CD which is **equal to any part** (*e.g.* AP) of AB.

In practice straight lines are made by drawing a line along the edge of a flat ruler, called a straight-edge, *i.e.* by making a copy of a line which we already believe to be straight. Straight-edges are often slightly warped; their accuracy should

be tested as follows: rule a line *ab*, then very close to it rule *a'b'* with the straight-edge reversed; the eye can then judge

whether the line *a'b'* would lie along *ab* when moved on to it, or whether there would be gaps between them.

Postulates.

Note.—A **postulate** is a statement that it is possible to perform some simple geometrical operation, and it is, in addition, a **demand** to be allowed to perform it without explaining.

We shall assume that a straight-edge can be used as follows:—

Postulate 1.—A straight line can be drawn from any point to any other point.

Postulate 2.—A finite line (*i.e.* a line "finished" at both ends) can be produced or continued at either end to any distance.

We shall also assume that a pair of compasses or of "dividers" (*i.e.* compasses with both points sharp) can be used as follows:—

Postulate 3.—From any straight line a part can be cut off or marked off equal to any shorter line.

Axioms.

Note.—An **axiom** is the statement of a fact which cannot be proved by means of any better-known facts, and which is so well known from experience that all reasonable people accept it as true.

We know from experience that a straight line ruled along one straight-edge is found to be straight if tested by any other

straight-edge with which we choose to compare it: hence all straight lines may be considered as copies of the same straight line (so far as *straightness* is concerned; of course all straight lines are *not of the same length*), and therefore the following statements must be true.

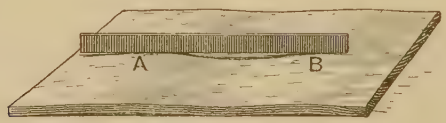
Axiom.—Two straight lines cannot enclose a space.

Axiom.—If two straight lines be made to coincide along part of their length, they cannot diverge (*i.e.* open out from each other) at any other part of their length.

Plane.

Definition.—A **plane** is a flat surface—that is, it is a surface such that if any two points on it be joined by a straight line, the line lies entirely on the surface.

This definition gives a practical test for a plane: if a straight-edge be applied to a surface, as in the figure at A and



B, and we can "see daylight" between the two, the surface is not flat, but if we can sweep the straight-edge about over the surface without any gaps appearing, then the surface is plane.

Any figure which lies in a single plane is called a **plane figure.**

Direction.

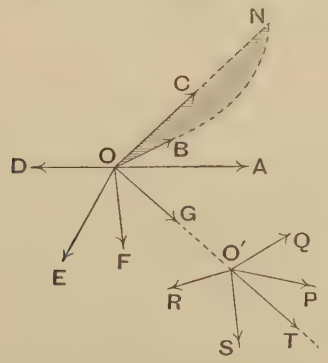
In the figure several straight lines are drawn from the point O: an arrowhead has been placed on each to show in which way the pen travelled while drawing it.

The different lines are drawn of different lengths, but we can lengthen any one, say OB, by placing a straight-edge along it and then producing or continuing the line along the straight-edge.

So long as we keep to the same line, OB, we cannot get on to OC or any other of the lines; for if we did find our pencil at N, a space, as shaded, would be enclosed between the lines OCN, OBN, and that would prove that one or the other of them had not been drawn straight.

Thus, after leaving O each of the lines OA, OB, etc., is perfectly distinct from the others, they all point in different ways, and we say that each marks a **different direction.**

OA and OD, which form parts of the same straight line AD, are said to be drawn in **opposite directions**, or often they are said to be drawn along the same line in **opposite senses.**



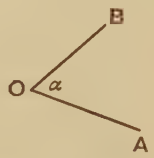
From O′ another set of lines is drawn. In the two sets, OG and O′T, which are **drawn in the same sense** along the **same line**, are certainly drawn in the **same direction.** As to the others, the reader will probably agree that OF and O′S point in the same direction, while OE and O′R do not. Also, if asked, "What *exactly* do you here mean by the same direction?" he will be at a loss for an answer. And the answer cannot be given till we have discussed "angles."

Each of the lines OA, OB, OC, etc., is called a **radius vector,** or more shortly a **radius.** They are so called since they radiate from the point O like rays of light, or like the spokes of a wheel (Latin, *radius;* French, *rayon*, "a spoke or ray").

Greek α, pronounced Alpha, corresponds to English *a*.
,, β, ,, Beta, ,, ,, *b*.

Angles.

If two straight lines OA, OB meet at a point O, in ordinary language O is called a corner; in geometry we say that OA



and OB **make** an **angle** with each other. O is called the **vertex**, and OA, OB the **arms** (or *sides*) of the angle.

The most convenient way to indicate the angle is by means of a sign placed "in the corner." It is an advantage to use always the same kind of letter to mark the same kind of thing: thus we use capitals, A, B, . . . to mark points; small letters, **a, b,** . . . or *a*, *b*, . . . to mark lines or lengths of lines; and we shall, wherever possible, use Greek letters, α, β, . . . to mark angles (a list of Greek letters used is given on p. vi.).

Sometimes, however, we indicate the angle by three letters, the middle letter being the one at the corner. The angle in the figure may be spoken of as follows:—

(1) The angle contained by OA, OB;
(2) The angle between OA, OB;
(3) The angle AOB, or $\angle$ AOB;
(4) The angle BOA, or $\angle$ BOA;
(5) The angle α, or $\angle\alpha$, or α.

So far we have said that OA and OB *make an angle*, but have not defined an angle.

Now, "angle" has at least two slightly different meanings.

First Meaning of Angle.

If we rule a straight line, OA, and then rule the line OB with the same straight-edge, the straight-edge may be turned round the point O in order to reach its new position: while turning, the direction in which it points is continually being changed.

The **angle** between OA and OB is the **amount of change of direction** or **amount of turning** required to pass from the position OA to the position OB.

The position OB may be reached by turning (Fig. 1) in the same way as the hands of a clock turn (*i.e.* clockwise), or by turning (Fig. 2) in the opposite way (*i.e.* anticlockwise).

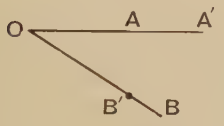


Fig. 1.

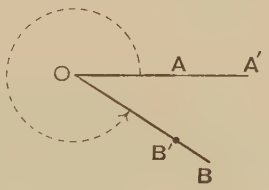


Fig. 2.

Hence two lines OA, OB "contain" two angles. As a rule, when an angle is spoken of, the one is meant which requires the less amount of turning.

Equality of Angles.

Two angles are **said to be equal** if each is produced by the same amount of turning.

But how are we to judge the "amount of turning"? First, notice that when the straight-edge is in position for ruling OA, we can draw a short line, OA, or a longer one, OA′; and when we have turned it into position for ruling OB, we can draw a short line OB′ just as well as the longer one OB. Hence the **amount of turning** depends only on the **direction of the lines**, and **not on their length:** the angle AOB is identically the same as the angle A′OB′.

How shall we **test the equality of** α and β? Make a copy of α, say by tracing on tracing-paper, or by cutting out a piece of paper to fit it; place the copy on β, so that O covers

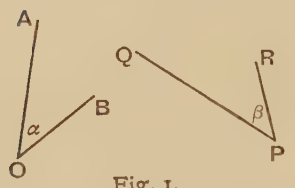


Fig. 1.

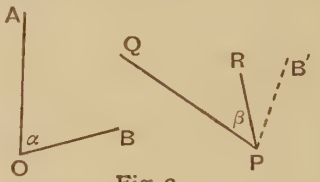


Fig. 2.

P and OA **lies along** PQ. **If** OB **then lies along** PR, **we say that the angles are equal.**

But if OB lies in a direction PB′, as in Fig. 2, we say that α is greater than β.

Second Meaning of Angle.

Take a pair of compasses; open them.

The **angle** between the "legs" of the compass is the "**wideness**" to which the compasses are opened.

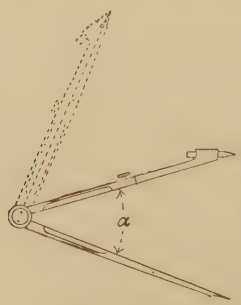


Fig. 1.

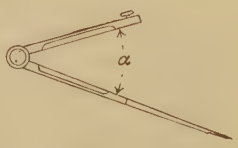


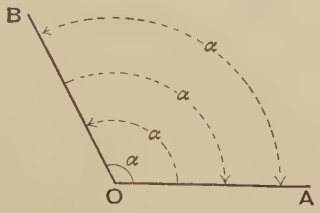
Fig. 2.

The distinction between this idea of an angle and the preceding is that we think only of the final appearance of the compasses, and not of how the legs moved during the opening: they may at first have been in the dotted position, and both of

them have moved; or either of them may have been fixed while the other turned—the result in each case is the same angle α.

Here, also, we see that the length of the lines does not matter: α in Fig. 2 with the pencil fitting taken away is the same as α in Fig. 1—both are equally "wide."

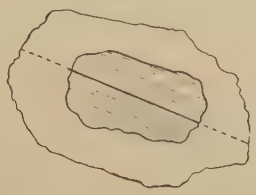
In subsequent figures, to mark an angle, we shall place a letter α, β, etc., either close to the letter O where the "arms"



of the angle meet, or in any convenient position even at some distance from O; and dotted lines with small arrowheads at their ends will be used to indicate the extent of the angle.

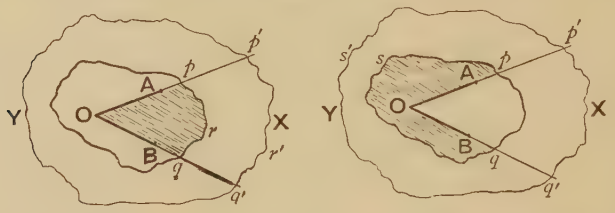
Third Meaning of Angle.

If a straight line be drawn across a flat piece of paper, it divides the surface into two portions, one on either side of it; and these portions are not necessarily equal. Now imagine



the size of the paper to be increased, and the straight line produced to the new boundary; the line still divides the paper into two portions. Continue the process without limit; we then have a boundless plane traversed by a straight line and divided into two portions by it.

Next draw two straight lines **OA**, **OB** on a flat piece of paper, and produce them to the boundary: they divide the paper into



two portions, the first (shaded in the left-hand figure) between the lines, the second (shaded in the right-hand figure) not between them.

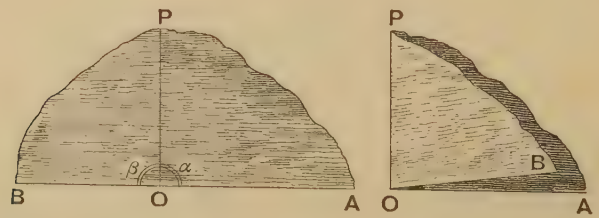
As before, imagine the paper increased and the lines produced to meet the new boundary at $p'q'$, and continue the process without limit.

One of the angles contained by **OA**, **OB** is the portion of plane space between **OA***pp'* . . . and **OB***qq'* . . ., considered as definitely bounded by them on two sides, but as either boundless towards **X**, or as having a vague boundary *prq* or *p'r'q'* drawn anywhere and anyhow we please from one line to the other.

The other angle contained by **OA**, **OB** is the portion of plane space *not* between **OA***pp'* and **OB***qq'*, and is boundless towards **Y**, or has any vague boundary *psq*, or *p's'q'*.

Right Angles.

A straight line such as **OP**, which stands straight out from another line **AB**, is said to be perpendicular to it, or at right



angles to it, and each of the angles α, β is called a right angle.

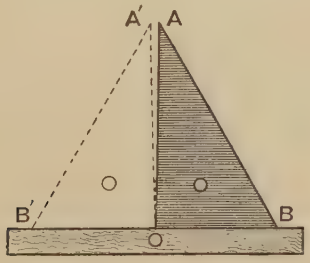
If AB is the edge of a piece of paper, and we wish to obtain a line standing out at right angles at O, we fold the paper at O and try to make OB lie exactly along OA.

If we succeed, then the angle β on the left of the crease OP has been fitted on to the angle α on the right of it, and therefore β equals α. The equality of these adjacent angles is the test of their being right angles.

Definition.—If a straight line OP meet a straight line AB at O, and makes the **adjacent angles** α, β **equal**, then α and β are called **right angles**.

α, β are called **adjacent angles**, because they lie next each other, having a common arm OP.

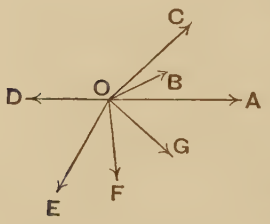
In practice right angles are made, not by folding, but by the constructions given in Chapter V., or by copying the angle of a "set-square." Common set-squares are not accurate, and should be tested as follows:—



Place the set-square AOB against a straight-edge and draw OA; hold the straight-edge firm, turn the set-square over into the position A′OB′, and draw OA′.

If OA′ coincide with OA, the square is accurate; if it lie as in the figure, then the angle of the square is less than the angle AOB′, so that OA does not make the adjacent angles equal, hence the "square" is inaccurate.

If a straight line, such as OA in figure below, turn round the fixed point O, and pass in succession through the positions OB, OC . . . OG until it comes back to the position OA, we say that it has made a **complete revolution** (or shortly "a revolution"). For example, the minute-hand of a clock makes one complete revolution in 60 minutes, and the hour hand makes one revolution in 12 hours.



In passing from OA to OD, which is opposite to OA, we say that the line has made half a revolution, *e.g.* the minute-hand makes a half-revolution in 30 minutes, say, while it is passing from the mark X to IIII. Are we justified in calling this a "half"-revolution? Yes. For if the paper be folded along the line AOD, the angle which is the "turning from OA to OD" is placed upon the angle which is the "turning from OD to OA," in the manner demanded in our definition of equal angles.

Again, referring to the figure illustrating right angles, we see that each right angle is half of half a revolution, *i.e.* a right angle is a quarter of a revolution.

If a right angle be divided into 90 equal parts, each part is called a **degree**; also the sixtieth part of a degree is called a **minute** (of angle), and the sixtieth of a minute is called a **second** (of angle).

Hence **a revolution is a turning through four right angles**, *i.e.* 360°; and **half a revolution is a turning through two right angles**, *i.e.* 180°.

Definitions.—An angle **less than a right** ∠ is called **acute.**

An angle **greater than a right** ∠, and less than two right ∠s, is called **obtuse.**

An angle **greater than two right** ∠**s** is often called **reflex.**

Exercises.

[In each answer state whether the ∠ is *acute*, *obtuse*, or *reflex*.]

1. How many degrees does the minute-hand of a clock turn through in 20 minutes? and how many in 5 minutes?

2. If one arm of a sign-post point N. and another N.E., what angle do they make with each other?

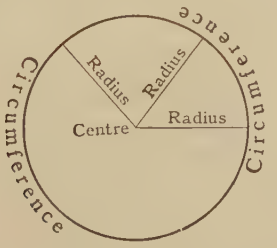
3. One ship sails N.N.W. and another S.W. from the same place: what angle do their courses make with each other?

4. A wheel makes 14 revolutions in a minute: how many degrees does each spoke turn through in a second?

5. A wheel makes 150 revolutions in a minute: how many degrees does each spoke turn through in one-hundredth of a second?

Circles.

Definition.—A circle is a plane figure bounded by a curved line called its **circumference**; it is such that all



radii drawn from a fixed point within it to the circumference are **equal.**

The fixed point is called the **centre.**

Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called **a diameter.**

Postulate.—A circle can be described (*i.e.* drawn) with any point as centre and with radius of any required length, *i.e.* with radius equal to any given line.

Greek α, pronounced Alpha ; β, pronounced Beta.

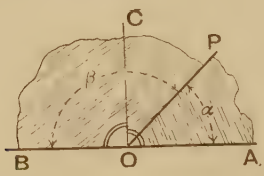
CHAPTER II.

CONCERNING ANGLES AND PARALLELS.

PROP. 1. [Euclid I. 13.]

The two angles which one straight line makes with another straight line on the same side of it are together equal to two right angles.

Let OP make the $\angle$s α, β with AB.
Required to prove that $\alpha + \beta = 2$ rt. $\angle$s.



Proof.—If a line start from the position OA and rotate round O to the position OB, it first sweeps through the angle α and then through the angle β, and in so doing it makes half a revolution.

$$\therefore \alpha + \beta = \tfrac{1}{2} \text{ revolution} = 2 \text{ rt. } \angle\text{s}$$

Another proof.—(1) If $\alpha = \beta$, each *is* a right angle.

$$\therefore \text{their sum} = 2 \text{ rt. } \angle\text{s}$$

(2) If not, suppose OC is perpendicular to AB.

Then α and β together fill all the space on one side of AB,
and the rt. $\angle$s also „ „ the same „
$\therefore \alpha + \beta = 2$ rt. $\angle$s (Definition of equality of angles).

Definition.—A **corollary** to a proposition is an additional theorem, whose truth is easily seen when the proposition has been proved.

Corollary 1.—From the same point on a line only one line can be drawn on the same side of it and at right angles to it.

For if we could imagine both OP and OC to be at rt. $\angle$s to AOB, then α, β would be equal $\angle$s.

But α is less than one of the equal rt. $\angle$s made by OC, and β is greater than the other, hence α and β cannot be equal.

Corollary 2.—All right angles are equal.

For if there were another figure with a line O′C′ at rt. $\angle$s to A′O′B′, it could be placed on this figure, so that the right angles of one figure fitted the right angles of the other.

Definition.—If the **sum of two angles** be a **right angle**, each is called the **complement** of the other (*i.e.* the *complement* is the $\angle$ necessary to *complete* a rt. $\angle$).

EXAMPLE.—42° is the complement of 48°, since 42° + 48° = 90°.

Definition.—If the **sum of two angles** be **two right angles**, each is called the **supplement** of the other (*i.e.* the *supplement* is the $\angle$ *supplied* to make 2 rt. $\angle$s).

EXAMPLE.—123° 51′ is the supplement of 56° 9′, since—
123° 51′ + 56° 9′ = 180°.

EXERCISES.

1. In figure of Prop. 1, what is the value of β for the following values of α: 31°, 68°, 72° 23′, 120°, 152°?

2. Write down the supplements of the following $\angle$s: 16° 30′, 58°, 117° 45′, 175°.

3. Write down the complements of the following $\angle$s: 15°, 81°, 23° 27′.

Converse of a Proposition.

A theorem generally consists of the statement that "if some thing or things be true," then "some other things must be true also;" *i.e.* it consists of two clauses, the first of which states certain conditions under which the other clause becomes true.

The first clause is called the **hypothesis** (a Greek word meaning "a placing under") or the **supposition** (from Latin *sub*, "under;" *positum*, "placed"), for it is the foundation on which the argument is built. The other clause, which is the consequence of the first, is said to **follow from it**, and is called the **conclusion**.

Here is a theorem—not geometrical—

(hypothesis) "if a man be beheaded"

(conclusion) "he dies."

Here is another theorem formed from it, by interchanging the hypothesis and the conclusion—

(hypothesis) "if a man die"

(conclusion) "he is beheaded."

Each of such a pair of theorems is said to be the **converse** of the other. The first of the above pair is probably true in all cases; but its converse, though true in some instances, is certainly not always true.

Here is a geometrical theorem—

(hypothesis) "If two ⊙s intersect"

(conclusion) "they have different centres."

The converse theorem is—

(hypothesis) "If two ⊙s have different centres"

(conclusion) "they intersect."

This converse is not true, for if we draw two small ⊙s with centres far apart, they obviously do not intersect.

These instances show that, as a rule, when a theorem has been proved, we ought not to assume without proof that its converse is also true.

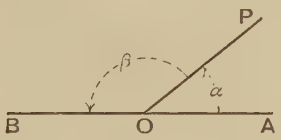
PROP. 2 (CONVERSE OF PROP. 1). **[Euclid I. 14.]**

If from a point on a line there be drawn two other straight lines on opposite sides of it, making the two adjacent angles together equal to two right angles, then these two lines are parts of the same straight line.

Let OA, OB be two straight lines making $\angle$s α, β with OP on opposite sides of it.

Given that $\alpha + \beta = 2$ rt. $\angle$s.

Required to prove that AOB is a straight line.



Proof.—Let a straight line, rotating anticlockwise round O, start from the position OA and turn through the angle α, let it then continue to turn through the angle β. It now lies along OB; but it has turned through an $\angle$ $(\alpha + \beta)$, which is by supposition 2 rt. $\angle$s.

$\therefore$ It has made half a revolution;

$\therefore$ it points in the opposite direction to OA,

$\therefore$ OB, along which it lies, points in the opposite direction to OA,

i.e. AOB is a straight line.

We see that Prop. 2 is the converse of Prop. 1, since we have—

in Prop. 1, hypothesis, "AOB is a straight line;"
conclusion, "$\alpha + \beta = 2$ rt. $\angle$s."

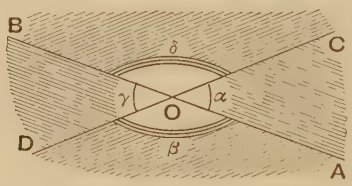
in Prop. 2, hypothesis, "$\alpha + \beta = 2$ rt. $\angle$s;"
conclusion, "AOB is a straight line."

Greek γ, pronounced Gamma ; δ, pronounced Delta.

Prop. 3. [Euclid I. 15.]

If two straight lines cut one another, the vertically opposite angles are equal.

Let AOB, COD be two straight lines cutting at O, and making the four angles α, β, γ, δ.



Required to prove : $\alpha = \gamma$ and $\beta = \delta$.

Proof.—Place a straight-edge along AOB ; turn it anti-clockwise round O into the position COD.

α is $\angle$ turned through
γ „ „ „
$\Big\}$ $\therefore \alpha = \gamma$

Similarly $\beta = \delta$.

Another proof.—

OA meets COD $\therefore \alpha + \beta = 2$ rt. $\angle$s (Prop. 1)
OD „ AOB $\therefore \gamma + \beta = 2$ rt. $\angle$s
$\therefore \alpha + \beta = \gamma + \beta$ ($\because$ each sum = the same thing)
$\therefore \alpha = \gamma$ (taking β from each sum)

Similarly $\beta = \delta$.

Definition.—A **quadrilateral** is a plane figure bounded by **four straight** lines.

Definition.—A **polygon** is a plane figure bounded by more than four straight lines.

Definition.—A quadrilateral or a polygon is said to be

convex, if when all its sides are produced (either way) the produced parts lie altogether outside the polygon.

Definition.—A polygon is **regular** if all its sides are equal and all its angles equal.

In Prop. 3 we assumed two of the following axioms:—

Axiom.—Things which are equal to the same thing or to equal things are equal to each other.

Axiom.—If the same amount be added to things which are equal, the sums are equal.

Axiom.—If the same amount be taken from things which are equal, the remainders are equal.

Exercises.

1. A man standing at **O** can see two spires **A**, **B** and a light-house **P**: he measures the ∠ between **OA** and **OP**—it is 61°; and also the ∠ between **OB** and **OP**—it is 118°. Is the man on the straight line from **A** to **B**?

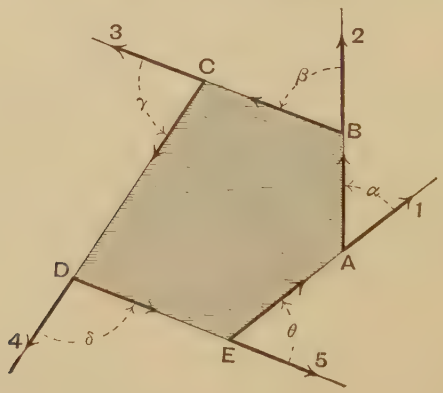
2. In figure of Prop. 3, if **OP**, **OQ**, **OR**, **OS** be drawn bisecting α, β, γ, δ, prove (1) that **POR** and **QOS** are straight lines; (2) that **OP**, **OQ** are at rt. ∠s.

Greek θ, pronounced Theta.

PROP. 4.

If each side of a convex polygon be produced in the same way, the sum of the exterior angles thus formed is equal to four right angles.

Let $\alpha, \beta, \gamma, \delta, \theta$ be the exterior $\angle$s formed by producing the sides of polygon **ABCDE** in the same way.



Required to prove: $\alpha + \beta + \gamma + \delta + \theta = 4$ rt. $\angle$s.

Proof.—Consider an arrow at **A** in position 1, pointing along **EA** produced.

Turn it through the exterior $\angle \alpha$; it then points along **AB**.

Slide it along **AB** to position 2, without turning.

Turn it through exterior $\angle \beta$; it then points along **BC**.

Proceed in this way till position 5 is reached, then turn the arrow through exterior $\angle \theta$; it now points along **EA**, and may be slid along **EA** to its original position 1 without further turning.

During these movements the arrow has turned in the same way through each of the $\angle$s $\alpha, \beta, \gamma, \delta, \theta$, in succession;

also it has made one complete revolution, *i.e.* it has turned through 4 rt. ∠s.

∴ the sum of the exterior angles = 4 rt. ∠s.

NOTE.—This proposition is also true for a triangle and for a convex quadrilateral.

Corollary.—At each corner, int^{r}. ∠ + ext^{r}. ∠ = 2 rt. ∠s.

∴ sum of int^{r}. ∠s + sum of ext^{r}. ∠s
= twice as many rt. ∠s as there are corners;

∴ Sum of interior ∠s + 4 rt. ∠s
= twice as many rt. ∠s as figure has sides.

NOTE.—A polygon with 5 sides is called a pentagon.
,, ,, 6 ,, ,, hexagon.
,, ,, 7 ,, ,, heptagon.
,, ,, 8 ,, ,, an octagon.
,, ,, 10 ,, ,, a decagon.

EXERCISES.

1. What is the value of the exterior angle formed by producing one of the sides of a regular octagon?

2. How many degrees are there in each exterior ∠ (1) of a regular pentagon; (2) of a regular hexagon?

3. How many degrees are there in each interior ∠ (1) of a regular hexagon; (2) of a regular decagon?

4. Find the number of degrees in an angle of a regular figure of eighteen sides.

5. The average size of the interior angles of a convex polygon is 140°: how many sides has it?

6. Three of the exterior ∠s of a quadrilateral are 81°, 111°, 148°: what is the fourth exterior ∠?

Greek γ, pronounced Gamma; δ, pronounced Delta.

PROP. 5. [Euclid I. 32.]

The three angles of a triangle are together equal to two right angles.

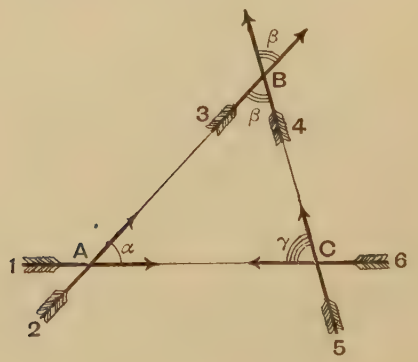
Let ABC be the triangle.

Required to prove: $\alpha + \beta + \gamma = 2$ rt. $\angle$s.

Proof.—Consider an arrow at A in the position 1, pointing along AC.

Turn it round A into position 2, pointing along AB.

It has swept through α.



Slide it to B, position 3, without turning.

Turn it round B into position 4, pointing along CB.

In doing this it sweeps through β.

Slide it backwards to C, position 5, without turning.

Turn it round C into position 6, pointing along CA.

In doing this it sweeps through γ.

The arrow is now pointing in exactly the reverse direction

to 1, therefore it has made a half-revolution, *i.e.* it has turned through two right angles.

Also at each corner it turned in the same way (anticlockwise), hence altogether it has turned through the sum of the angles α, β, γ.

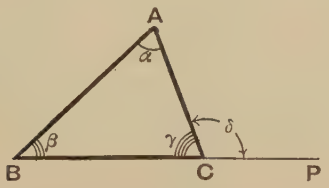
$$\therefore \alpha + \beta + \gamma = 2 \text{ rt. } \angle s$$

Corollary 1.—Since the sum of the three angles is two right angles,

$\therefore$ The **sum of any two angles** of a triangle is **less than two right angles.** [Euclid I. 17.]

Corollary 2.—Produce any side BC to P, so as to make an *exterior* angle δ.

$$\left.\begin{array}{r} \text{Then } \gamma + \delta = 2 \text{ rt. } \angle s \\ \text{But } \gamma + (\alpha + \beta) = 2 \text{ rt. } \angle s \end{array}\right\} \therefore \gamma + \delta = \gamma + (\alpha + \beta)$$
$$\therefore \delta = \alpha + \beta$$



That is, **any exterior angle** of a triangle is **equal to the sum of the two interior non-adjacent angles.** [I. 32.]

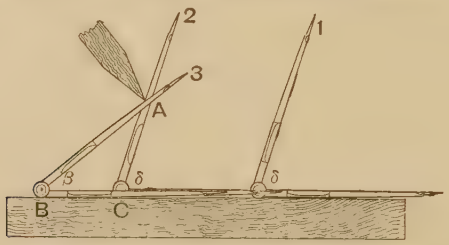
Corollary 3.—Any exterior angle is greater than either of the interior non-adjacent angles. [I. 16.]

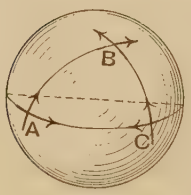
The truth of this may also be seen in a different way.

Open a pair of compasses to any angle δ; place them in position 1, with one leg along a straight-edge. Slide the compasses along the straight-edge to position 2, where the other leg comes up against the wedge-shaped "stop" placed at A: so far the angle δ has remained unchanged, but the compasses cannot be pushed further without partly closing the

compasses. When pushed into position 3, they make the angle β, which is the interior $\angle$ of $\triangle$ **ABC**; hence—

δ, exterior to $\triangle$, is greater than β, interior to $\triangle$





NOTE 1.—The essence of the proofs of Props. 4 and 5 is the fact that there is no change of direction while an arrow is sliding along a side of a polygon or of a triangle (see Introduction, p. 13). The constancy of direction depends on the fact that the polygons and triangles, which we are dealing with, are plane (*i.e.* flat) figures, and that their sides are straight. The figure shows how the proof would break down if applied to a "spherical triangle" (which consists of three circular lines drawn on a sphere): here the arrow is continually changing its direction as it slides along **AB**, and **BC**, and **CA**.

NOTE 2.—Prop. 5 is a particular case of corollary to Prop. 4.

EXERCISES.

[In the following, α, β γ are the $\angle$s of a $\triangle$; and δ is the $\angle$ exterior to γ, as in figure of Prop. 5, Cor. 2.]

1. Given $\alpha = 82°$, $\beta = 63°$, how many degrees are there in γ.
2. „ $\alpha = 48°$, $\beta = 59°$, calculate the value of δ.
3. „ $\delta = 113°$, $\beta = 78°$, „ „ α.
4. „ $\delta = 98° 48'$, $\alpha = 57° 53'$, calculate the value of β.
5. If $\alpha = 2\gamma$ and $\beta = 2\gamma$, find the value of each.
6. If $\beta = 3\alpha$ and $\gamma = 4\alpha$, find the value of each.
7. Two straight railways starting from a junction make an angle of 72° with each other; one crosses a straight canal at right $\angle$s: at what angle does the other cross it?
8. Two steamers start from different piers on a straight coast, one at an $\angle$ of 50° with the coast, the other at an $\angle$ of 72°: sketch their courses. If they cross, at what $\angle$ do they cross? [This

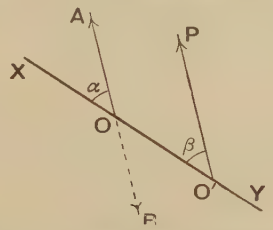
question is ambiguous ; two distinct sketches may be made, each giving a correct answer.]

9. **AO, BO** bisect the angles **A, B** of the triangle **ABC.** Prove that the angles **OAB, OBA** are together less than a right angle.

10. Within a △ **ABC** a point **O** is taken, so that ∠s **AOB, AOC** are equal to the exterior ∠s of △ at **C** and **B** : prove that ∠ **BOC** is equal to the exterior ∠ at **A.**

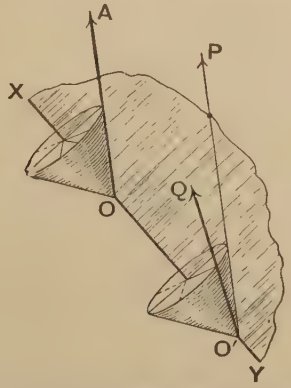
Direction of Lines.

Lines **OA, O′P** drawn (in the same plane) on the same side of **XY** from points **O, O′** in it are said to be **drawn in the**



same direction, if **the angle** α which **OA** makes with **OX** is **equal to the angle** β which **O′P** makes with **O′X.**

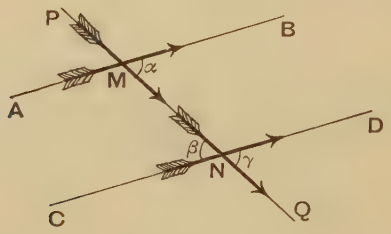
OB, which is drawn in the direction opposite to **OA** is also said to be drawn in the direction opposite to **O′P.**



Note.—The words "in the same plane" are used to make sure that the lines are not drawn like **OA, O′Q** in the figure.

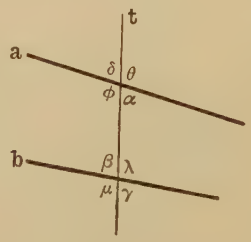
Greek λ, pronounced Lambda ; μ, pronounced Mu.
θ, ,, Theta ; φ, ,, Phi.

The idea of direction may also be expressed in a slightly different way. At **M** and **N** arrows are drawn pointing in



the same direction along **PQ**. If they be then **turned** in the same way **through equal angles** α, γ, they are still said to **point in the same direction**.

If a straight line **t** cross two other straight lines **a**, **b**, it makes four angles, α, δ, θ, φ, where it cuts **a**, and four angles, β, γ, λ, μ, where it cuts **b**.



α, β, λ, φ between **a** and **b** are called **interior** angles.
γ, δ, μ, θ not between ,, ,, **exterior** ,,
The interior ∠ α is **alternate** to β on opposite side of **t**.
,, ,, λ ,, ,, φ ,, ,, ,,

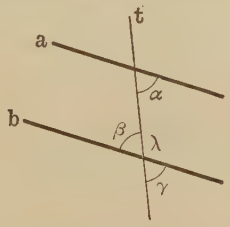
The interior $\angle\ \alpha$ and exterior $\angle\ \gamma$, which are one below a and one below b, and both on the same side of t, are called corresponding $\angle$s.

Also θ corresponds to λ (one above **a** and one above **b**).
δ ,, ,, β (,, **a** ,, ,, **b**).
ϕ ,, ,, μ (one below **a** and one below **b**).

IMPORTANT NOTE.—For alternative definition of parallel lines, and alternatives to Props. 6 and 7, see pp. 116 *b*, *c*, *d*, *e*.

Definition.—Two straight lines in the same plane are said to be **parallel** if **one of the interior angles** made by a transverse straight line **be equal to the corresponding exterior angle**, *i.e.* if the lines be drawn in the same or in opposite directions.

Thus in the figure **a** is parallel to **b**, if $\alpha = \gamma$.



Now, $\gamma = \beta$ (vertically opposite $\angle$s), and $\lambda + \beta = 2$ rt. $\angle$s.
$\therefore\ \alpha = \gamma$ (1) if $\alpha = \beta$, or (2) if $\lambda + \alpha = 2$ rt. $\angle$s.

$\therefore$ **a** is parallel to **b**—
(1) if $\alpha = \beta$,
i.e. if an interior angle = the alternate interior $\angle$.
(2) if $\lambda + \alpha = 2$ rt. $\angle$s,
i.e. if sum of interior $\angle$s on same side = 2 rt. $\angle$s.

The symbol ‖ stands for the word "parallel."

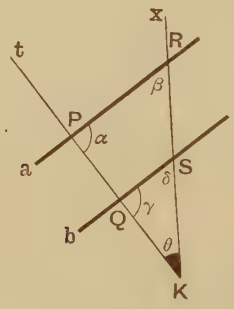
Prop. 6.

Two straight lines which are parallel when tested by one transverse line are parallel when tested—

(1) *by any transverse line which meets the first;*

(2) *by any transverse line which is not known to meet the first.*

Let **a**, **b** make equal ∠s α, γ on the same side of the transverse line **t**.



(1) Required to prove: a, b are ‖ as tested by the transverse line **x**, which meets **t** at **K**.

Proof.—$\alpha + \theta + \beta = 2$ rt. ∠s (∠s of △ PKR)

$\gamma + \theta + \delta = 2$ rt. ∠s (∠s of △ QKS)

$\therefore \alpha + \theta + \beta = \gamma + \theta + \delta$

$\therefore \alpha + \beta = \gamma + \delta$

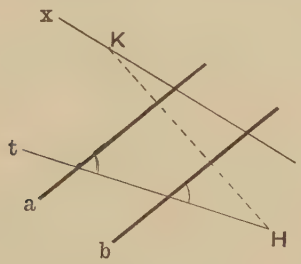
But $\alpha = \gamma$ (hypothesis)

$\therefore \beta = \delta$ (by subtraction)

i.e. **a**, **b** are ‖ as tested by **x**.

(2) Required to prove: a, b are || as tested by the transverse line **x**, which does not meet **t**.

Proof.—Take any point **H** in t, below b.
„ „ **K** in **x**, above **a**.

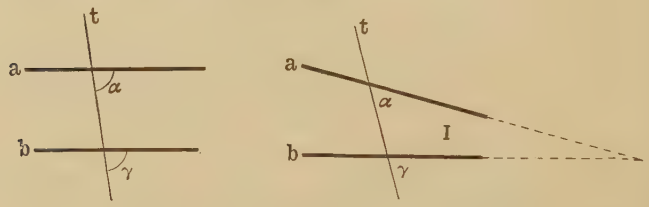


Transverse line **HK** meets **t**;
∴ **a**, b are ∦ when tested by **HK** (by first part of Prop.).
Transverse line **x** meets **HK**;
∴ **a**, b are || when tested by **x** (by first part of Prop.).

PROP. 7.

Two parallel straight lines do not meet when produced.

Let a, b be || straight lines, making equal ∠s α, γ with t.
Required to prove: a, b cannot meet.



Proof.—One of the following statements must be true, and only one can be true :—

either (1) a, b do meet,
or (2) „ do not meet.

If a, b were to meet when produced, say, to the right, then I would be a △; and γ (exterior ∠) would be greater than α (interior non-adjacent ∠); but it is not.

∴ a, b do not meet towards the right
similarly a, b „ „ „ left

∴ (1) is not true.
∴ (2) must be true,

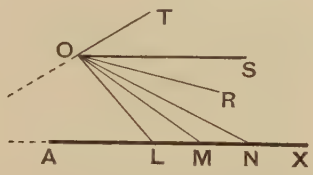
i.e. a, b do not meet when produced.

Corollary.—From a point outside a straight line only one straight line can be drawn to it at right angles.

For any two lines which are at right angles to the same line are parallel, and if both were drawn from the same point, they would be a pair of intersecting parallel lines.

Consider a straight line **AX**, and draw a set of lines **OL**, **OM**, etc., through the same point **O**.

In drawing **OL** the straight-edge is inclined fairly steeply to **AX**. Then to draw **OM**, **ON**, etc., it must be rotated so that its inclination decreases, and so that it points more nearly



in the same direction as **AX**. At the same time the points of intersection, **L, M, N,** recede from **A.**

OR does not meet **AX**, but we see that it would if both were produced far enough.

OS, which points in the same direction as **AX** and is parallel to it, does not meet it however far produced.

OT would meet **AX**, if both were produced backward.

We shall assume, as evident from experience, that every line through **O**, except **OS**, meets **AX**, if both be produced far enough one way or the other; that is, we assume as an **axiom—**

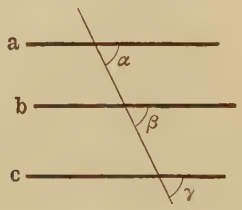
All straight lines which are in the same plane and which are not parallel meet if produced far enough.

Prop. 8. [Euclid I. 30.]

Lines which are parallel to the same line are parallel to each other.

Let **a**, **b** be each || to **c**.

Required to prove: **a** is || to **b**.



Proof.—Let any transverse line make corresponding ∠s α, β, γ with **a**, **b**, **c**.

$\alpha = \gamma$ ($\because$ **a**, **c** are ||)

$\beta = \gamma$ ($\because$ **b**, **c** are ||)

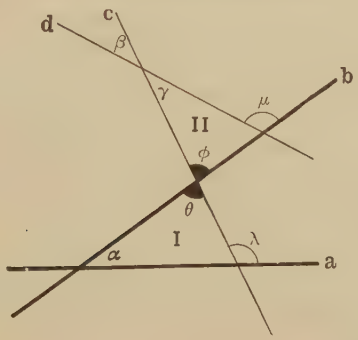
$\therefore \alpha = \beta$

$\therefore$ **a**, **b** are ||

Greek λ, pronounced Lambda ; μ, pronounced Mu ; φ, pronounced Phi.

Additional Proposition.

If straight lines **c**, **d** *make equal angles* λ, μ *with two intersecting lines* **a**, **b** *on corresponding sides of them, then the* ∠ α *between* **a**, **b** *is equal to the corresponding* ∠ β *between* **c**, **d**.



Proof.—

λ is exterior to △ I ∴ $\lambda = \alpha + \theta$ (interior non-adjacent ∠s)

μ „ „ △ II ∴ $\mu = \gamma + \phi$ („ „)

But $\lambda = \mu$

$\therefore \alpha + \theta = \gamma + \phi$

Also $\theta = \phi$ (vertically opposite ∠s)

$\therefore \alpha = \gamma$

$\therefore \alpha = \beta$

Note 1.—The student should try to picture **c**, **d** merely as new positions of **a**, **b** ; thus **a** may turn through λ into position **c**, while **b** turns through the equal ∠ μ into position **d**. During this movement the direction of each has changed by the same amount ; hence β, the ∠ between their final directions, must be the same as α, the ∠ between their initial directions.

Note 2.—This proposition is of frequent use in the particular case when **c**, **d** are ⊥ to **a**, **b** ; *i.e.* when λ, μ are rt. ∠s. This case is easier to understand ; the student should draw a figure for it.

Important Exercise.—If a, b be a pair of intersecting straight lines, c a line || to a, and d a line || to b, prove that $\angle\ \alpha$ between a, b = the corresponding $\angle\ \beta$ between c, d.

EXERCISES.

1. If one $\angle$ of a $\triangle$ be equal to the sum of the other two, prove that the $\angle$ is a rt. $\angle$.

2. If two $\angle$s of one $\triangle$ be equal to two $\angle$s of another $\triangle$, prove that their third $\angle$s are also equal.

3. Straight lines ARB and PRQ cut each other at the point R; join AP, QB. If the two angles RAP, RQB, when added to one of the angles at R, together make up a right angle, prove that AP and QB when produced will cut each other at right angles.

4. Prove that an interior angle of a regular pentagon is three times as great as an exterior angle of a regular decagon.

5. ABCDE is a regular pentagon, and ABC′D′E′F′ a regular hexagon, described on the same side of AB: prove that the angle CBC′ is one-tenth of the angle ABC′.

6. Within $\triangle$ ABC straight lines AD, BE, CF are drawn making with AB, BC, CA respectively the $\angle$s DAB, EBC, FCA equal to each other: show that if AD, BE, CF do not meet in a point, they form a $\triangle$ whose $\angle$s are equal to those of $\triangle$ ABC.

7. Prove that the sum of two $\angle$s of an equilateral $\triangle$ is equal to one of the $\angle$s of an equiangular hexagon.

8. Prove that the sum of the angles of a pentagon is six right angles.

9. How many sides has a regular polygon, each of whose interior angles is three times as great as each of its exterior angles?

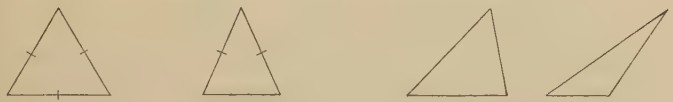
10. Prove that the angle contained by the bisectors of any two angles of a triangle is an obtuse angle.

CHAPTER III.

TRIANGLES.

Classification. — Triangles are arranged in classes as follows :—

(1) According to the number of equal sides.



Equilateral
3 sides equal

Isosceles
2 sides equal

Scalene
No sides equal

(2) According to angles.

Right angled
1 right angle

Obtuse angled
1 obtuse angle

Acute angled
3 acute angles

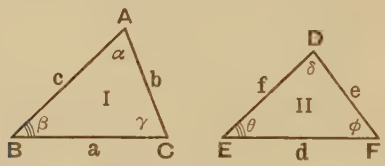
Definition.—The sum of the sides of any figure is called its **perimeter**; it is so called because it is the **distance measured all round** the figure.

NOTE.—The **base** of a △ is naturally its lowest side. Often it is convenient to turn a figure round to get a different view of it; this enables us to treat any side we please as base.

So also the **vertex** is naturally the highest corner; but very frequently each of the corners is called a vertex, the three being called the three vertices. In an isosceles triangle, however, the point where the two equal sides meet is always called *the* vertex, and the side opposite to it is *the* base.

The **height** (or **altitude**) of a △ is the length of the ⊥ from the vertex to the base. Thus in the figures of Prop. 12, if we regard **EF** as the base of △ **DEF, DP** is the height.

Parts of a Triangle.—The three $\angle$s α, β, γ, and the



three sides **a**, **b**, **c**, are called the six **parts** of the $\triangle$ ABC. [The points A, B, C at the corners are not "parts."]

The side **a** is said to be **opposite** to $\angle\alpha$, or to **subtend** the $\angle\alpha$
„ **b** „ „ „ β, „ „ „ β
„ **c** „ „ „ γ, „ „ „ γ

Equality of Triangles.—If the three sides and three angles of one triangle be equal to the three sides and three angles of another triangle, and if their areas be equal, the triangles are **equal in all respects**; for there are no other features of the triangles which we could wish to be equal. Two such triangles are also said to be **identically equal.**

Propositions 9, 12–15 are devoted to finding out how much information we must have concerning two triangles, before we know that they are identically equal.

Is it sufficient to know that side **a** in $\triangle$ I = side **d** in $\triangle$ II? Clearly not.

If we know, in addition, that β in $\triangle$ I = θ in $\triangle$ II, will that suffice? Again, clearly not.

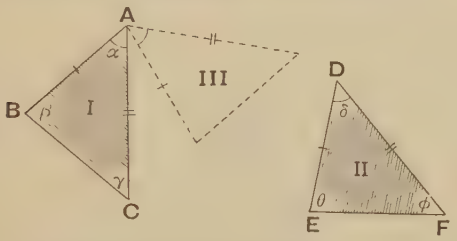
The student should ask himself as many questions as he can similar to the above, so as to discover the necessary tests of equality for himself before reading the propositions.

Definition.—Equality of figures. Two figures are **said to be equal in all respects,** or **identically equal,** or **congruent**, if one of them (or a copy of it), when placed on the other, can be **made to coincide** with it, *i.e.* to fit it exactly.

We shall use the symbol $\equiv$ to mean "is identically equal to."

Prop. 9. [Euclid I. 4.]

If two sides and the angle contained by them in one triangle be equal to two sides and the angle contained by them in another triangle, then the triangles are equal in all respects.



Let I, II, be the two △s, in which we are **given**—

$$\left.\begin{matrix} \mathsf{AB} = \mathsf{DE} \\ \mathsf{AC} = \mathsf{DF} \end{matrix}\right\} \text{and } \alpha = \delta$$

Required to prove: △ I ≡ △ II.

Proof.—Move the △ II up to the △ I, either by cutting it out or by copying it on tracing-paper, or, better still, by *imagining* it to be moved; place it so that the point D lies on the point A. II now lies in some such position as III; turn it round A so that DE lies along AB.

Now, DE is the same length as AB, and they are placed so as to begin at the same point A, therefore they end at the same point B.

Also, because $\alpha = \delta$, *i.e.* because the ∠s are equally "wide," DF must lie along AC.

Now, DF = AC, and they both begin at A, ∴ they both end at the same point C.

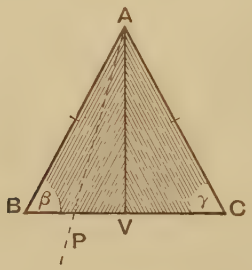
The two ends of EF now lie on the two ends of BC, and both are straight, therefore they coincide throughout their length. (Axiom.)

The △s I, II now coincide; ∴ they are equal in all respects. (Definition of "equality.")

Prop. 10. [Euclid I. 5.]

In an isosceles triangle, the angles opposite to the equal sides are equal.

Let ABC be the isosceles △, in which AB = AC.
Required to prove: $\beta = \gamma$.



[If we draw through A a line AP nearly coinciding with AB, and if it be then turned round A towards AC, it first makes a smaller ∠ with AB than with AC, finally it makes the smaller ∠ with AC; hence there must certainly be some intermediate position in which it makes equal ∠s with AB, AC, *i.e.* there must be some line which bisects ∠ BAC.]

Proof.—Suppose AV is the line which bisects ∠ BAC.

Let it meet BC at V.

Then in △s ABV, ACV—

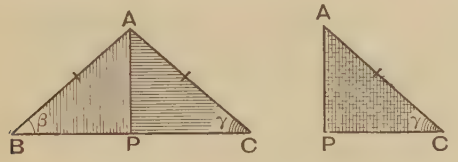
AB = AC (hypothesis)
AV = AV (∵ identical) } and ∠ BAV = ∠ CAV

∴ △s are equal in all respects.

∴ ∠ β, which is opposite to AV in △ ABV, is equal to ∠ γ „ „ „ „ △ ACV.

Second Proof.—By folding.
Given ABC is an isosceles △ having AB = AC.

Imagine the △ cut out; then fold it about a line AP through A, taking care that AB lies along AC.

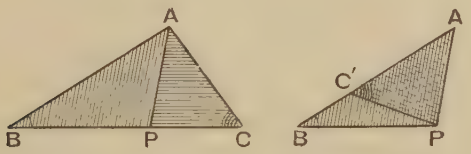


Unfolded. Folded.

Then B must come exactly on to C, since AB = AC.

Hence BP folds on to PC; and △ ABP folds on to and fits △ APC exactly, with β lying on γ.

$$\therefore \beta = \gamma$$



NOTE.—The second pair of figures shows how the above proof would have broken down, if one of the sides (*e.g.* AC) had been shorter than the other. Here the △ is folded over so that AC lies along AB, but since AC is shorter than AB it only reaches to C′; hence △ APC does not fold so as to fit on △ APB.

EXERCISES.

1. Prove that the triangle formed by joining the middle points of the sides of an isosceles triangle is isosceles.

2. Two isosceles △s CAB, DAB are on the same base AB: show that △ ACD ≡ △ BCD.

3. If on a common base and on opposite sides of it, two isosceles △s be described, the straight line joining their vertices cuts the base at rt. ∠s.

4. In the base BC of an isosceles △ ABC, points D and E are taken such that BD = CE. Prove that the ∠ BAD = ∠ CAE.

5. If the base of an isosceles triangle be produced, show that twice the exterior angle is equal to the vertical angle together with two right angles.

PROP. 11 (CONVERSE OF PROP. 10). [**Euclid I. 6.**]

If two angles of a triangle be equal, the triangle is isosceles.

Let **ABC** be the △, in which $\beta = \gamma$.
Required to prove: **AB = AC.**

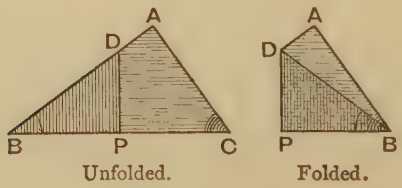
Proof.—Imagine the △ cut out; fold it by bringing the point **B** on to the point **C**, and then pressing flat.

If **P** be the point in **BC** where the crease comes, then **BP**, which is one of the arms of β, lies along **CP**, which is one of the arms of γ.

$\therefore$ **BA** lies along **CA** (for $\beta = \gamma$)

i.e. △ **APB** folds on to and fits △ **APC** exactly.

$\therefore$ **AB = AC**



NOTE.—The second pair of figures shows how the above proof would have broken down if one of the angles (*e.g.* β) had been less than the other.

Here also the △ is folded by bringing the point **B** on to **C**; but since β is less than γ, **BD** folds into the position shown instead of covering **CA**.

EXERCISES.

1. **O** is the middle point of a straight line **PQ**; **OR** is drawn perpendicular to **PQ**, and **PR**, **QR** are joined: prove that **PR** = **QR**.

2. The side **AC** of a △ **ABC** is bisected at **X**. **BX** is produced to **P** so that **XP** = **BP**, and **CP** is joined. Prove that △ **ABX** ≡ △ **CPX**.

Deduce that exterior ∠ of △ made by producing **BC** is greater than interior non-adjacent ∠ **A**.

NOTE.—This is Euclid's proof of Prop. 5, Cor. 2.

3. Perpendiculars are drawn to two sides of a △ **ABC**, from their middle points; they meet at **O**. Prove that **O** is equally distant from **A**, **B**, and **C**.

4. A quadrilateral **ABCD** has the sides **AB**, **CD** equal, and also the straight lines drawn from a certain point to its corners are all equal. Prove that its angles are equal in pairs.

5. A straight line **AB** is bisected at **O**, and from **O** a straight line **OC** = **OA** is drawn in any direction: prove that if **AC**, **CB** be joined, **ACB** is a rt. ∠.

6. If the lines **OB**, **OC** bisecting ∠s **ABC** and **ACB** of a △ **ABC** are equal, prove that △ **ABC** is isosceles.

7. If **ABC** be an isosceles △, and if the ∠s between the base and the sides produced be bisected by straight lines **BO**, **CO**, which meet at **O**, show that **OBC** is an isosceles △.

8. **ABC** is any acute ∠; **AB** is bisected at **D**, and at **K** in **BC** ∠ **DKB** is made = ∠ **DBK**: if **AK** be drawn, prove that it is ⊥ to **BC**.

9. If in the quadrilateral **ABCD**, **AD** = **BC**, and ∠ **ABC** = ∠ **BAD**, prove that ∠ **ADC** = ∠ **BCD**.

10. Prove that an equiangular triangle is also equilateral.

11. If a straight line which bisects an exterior angle of a triangle is parallel to the opposite side, show that the triangle is isosceles.

Prop. 12. [Euclid I. 8.]

If the three sides of one triangle be equal to the three sides of another triangle, then the triangles are equal in all respects.

Let ABC, DEF be the △s.

Given: AB = DE, AC = DF, BC = EF. **Required to prove:** △ ABC ≡ △ DEF.

Proof.—Move △ ABC up to DEF. Do this, as before, either by means of a tracing, or by cutting out, or make the movement in imagination only. Place ABC in the position EFA′, so that BC lies along EF—this is possible, for they are equal, and both are straight—and so that A′ and D are on opposite sides of EF.

Join A′D. It must cross EF, or EF produced, somewhere; for it is drawn from a point on one side of EF to a point on the other side of it. Let P be the point of intersection.

Then P must be (1) between E and F, as in Fig. 1;
or (2) at E or F, as in Fig. 2;
or (3) in EF produced, as in Fig. 3.

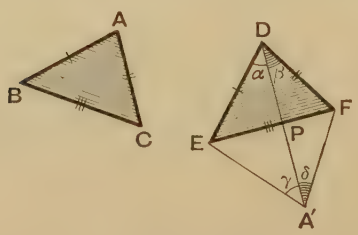


Fig. 1.—

EA′ = ED (hypothesis) ∴ △ EDA′ is isosceles ∴ $\alpha = \gamma$

FA′ = FD (,,) ∴ △ FDA′ ,, ∴ $\beta = \delta$

∴ ∠ EDF = $\alpha + \beta = \gamma + \delta$ = ∠ EA′F, which *is* ∠ BAC

We now know that in △s ABC, DEF—

$$\left.\begin{array}{l} AB = DE \\ AC = DF \\ \text{and } \angle BAC = \angle EDF \end{array}\right\} \therefore \triangle ABC \equiv \triangle DEF \text{ (Prop. 9)}$$

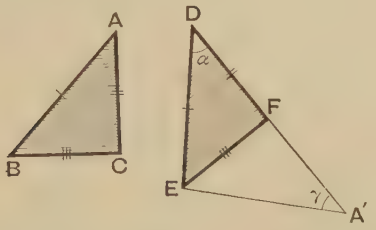


Fig. 2. $\angle EDF = \angle EA'F$ which *is* $\angle BAC$.
$\therefore \triangle BAC \equiv \triangle DEF$ (Prop. 9)

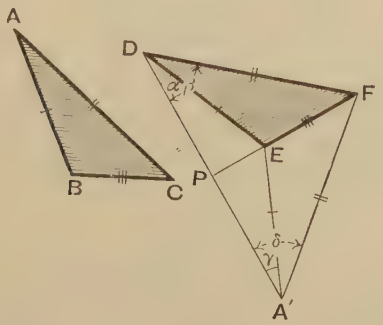


Fig. 3.—As before, $\beta = \delta$
and $\alpha = \gamma$
Subtract; $\therefore \beta - \alpha = \delta - \gamma$
i.e. $\angle EDF = \angle EA'F$, which *is* $\angle BAC$.
$\therefore$ as before, $\triangle ABC \equiv \triangle DEF$ (Prop. 9).

NOTE.—In the figures given for this proposition it was necessary to turn **ABC** over before fitting it on below **DEF.** Clearly figures might have been drawn in which this was unnecessary.

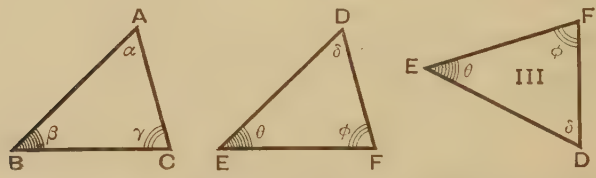
Prop. 13. [Euclid I. 26.]

If two angles and a side of one triangle be equal to two angles and the corresponding side of another triangle, the triangles are equal in all respects.

Let ABC, DEF be the △s.

Given: $\beta = \theta$, $\gamma = \phi$, BC = EF

Required to prove: △ ABC ≡ △ DEF.



Proof.—Place △ DEF on △ ABC, so that EF lies along BC. EF can be made to fit BC exactly, for they are equal, and both are straight.

Now, $\beta = \theta$, *i.e.* both ∠s are equally "wide,"

∴ ED lies along BA

Also $\gamma = \phi$, ∴ FD lies along CA

We have now proved that ED and FD coincide in direction with BA and CA.

∴ D, the point where ED, FD cut, must coincide with A, the point where BA, CA cut.

Hence △ EDF has been made to coincide with △ ABC.

∴ △ ABC ≡ △ DEF

Note 1.—If DEF be in the position III, it must be turned over before placing on ABC.

Note 2.—If we are given $\alpha = \delta$ instead of being given $\gamma = \phi$,

then $\alpha + \beta + \gamma = \delta + \theta + \phi$ (∵ each sum = 2 rt. ∠s)

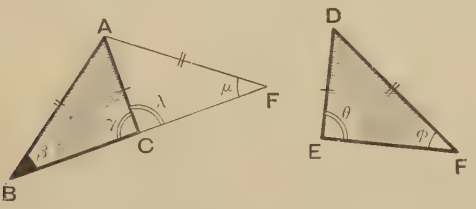
But $\alpha = \delta$ and $\beta = \theta$ ∴ $\gamma = \phi$

and the proof may be continued as before.

Prop. 14.

If two right-angled triangles have their hypotenuses equal, and also a side of one equal to a side of the other, the triangles are equal in all respects.

Let ABC, DEF be the △s.



Given: $\gamma = \text{rt. } \angle = \theta$
$AB = DF$
$AC = DE$

Required to prove: $\triangle ABC \equiv \triangle DEF$.

Proof.—Move △ DEF up to △ ABC; place it in the position ACF, so that DE lies along AC. This is possible, for they are equal, and both are straight.

γ and λ are rt. ∠s (∵ λ *is* θ in a new position)

∴ BCF is a straight line (Prop. 2)

∴ ABF is a triangle; also it is isosceles, since AB = DF

∴ $\beta = \mu$, *i.e.* $\beta = \phi$ (∵ μ *is* ϕ in a new position)

We now know that in △s ABC, DEF—

$$\left.\begin{array}{c}\beta = \phi \\ \gamma = \theta \\ AC = DE\end{array}\right\} \therefore \triangle ABC \equiv \triangle DEF \text{ (Prop. 13)}$$

Prop. 15.

If two sides of one triangle be equal to two sides of another triangle, and if the angles opposite to one pair of equal sides be equal, then the angles opposite to the other pair are **either** (*a*) *equal* **or** (*b*) *supplementary.*

Let ABC, DEF be the △s.

Given: $AB = DE$
$AC = DF$
$\beta = \theta$

Required to prove: either (*a*) $\gamma = \phi$,
or (*b*) $\gamma + \phi = 2$ rt. ∠s.

One of the following statements must be true :—

(1) $BC = EF$; (2) $BC < EF$; or (3) $BC > EF$

We will consider each in turn.

(1) Suppose $BC = EF$ (Fig. 1)

$AB = DE$
$BC = EF$
$\beta = \theta$ (included ∠s)
$\therefore \gamma = \phi$

$\therefore$ △ ABC ≡ △ DEF

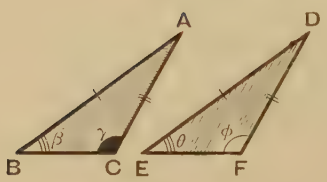


Fig. 1.

Fig 2.

(2) Suppose $BC < EF$ (Fig. 2).

Place △ ABC upon △ DEF so that AB lies along DE. AB can be made to fit DE exactly, for they are equal, and both are straight.

Now, $\beta = \theta$, *i.e.* both angles are opened equally widely.

$\therefore$ **BC** lies along **EF**

Also **BC** $<$ **EF**. $\therefore$ **C** takes a position **C′** between **E** and **F**.

Now **DFC′** is isosceles ($\because$ **DF** = **DC′**, which is new position of **AC**).

$$\therefore \mu = \phi$$

$$\text{But } \lambda + \mu = 2 \text{ rt. } \angle\text{s}$$

$$\therefore \lambda + \phi = 2 \text{ rt. } \angle\text{s}$$

$$i.e.\ \gamma + \phi = 2 \text{ rt. } \angle\text{s (since } \lambda \textit{ is } \gamma)$$

(3) Suppose **BC** $>$ **EF**

Place $\triangle$ **ABC** on $\triangle$ **DEF** as before.

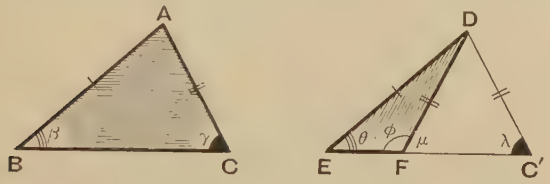


Fig. 3.

BC $>$ **EF** $\therefore$ **C** takes a position **C′** in **EF** produced.

$\triangle$ **DFC′** is isosceles; $\therefore \lambda = \mu$

$$\text{But } \mu + \phi = 2 \text{ rt. } \angle\text{s}$$

$$\therefore \lambda + \phi = 2 \text{ rt. } \angle\text{s}$$

$$i.e.\ \gamma + \phi = 2 \text{ rt. } \angle\text{s (since } \lambda \textit{ is } \gamma)$$

Hence in all cases either $\gamma = \phi$

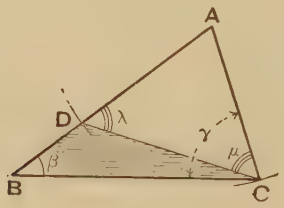
$$\text{or } \gamma + \phi = 2 \text{ rt. } \angle\text{s}$$

NOTE.—If desired, case (3) may be included in case (2) by stating, "If the bases be not equal, one must be the greater; then give the name **EF** to the greater." But it is probably easier for beginners to work the proposition as given.

Prop. 16. [Euclid I. 18.]

If one side of a triangle be greater than another, the angle opposite to the greater side is greater than the angle opposite to the smaller side.

Let ABC be the $\triangle$.
Given: AB $>$ AC.
Required to prove: $\gamma > \beta$.



Proof.—From AB cut off AD equal to AC. Join DC.
BDC is a $\triangle$, its side BD is produced to A.
$\therefore$ exterior $\angle$ λ is greater than interior non-adjacent $\angle$ β.
$\triangle$ ADC is isosceles; $\therefore$ $\mu = \lambda$

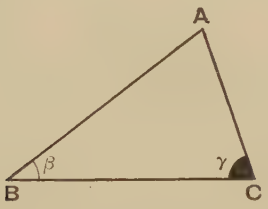
$$\left.\begin{array}{l} \therefore \mu > \beta \\ \text{But } \gamma > \mu \end{array}\right\} \therefore \gamma > \beta$$

Prop. 17. [Euclid I. 19.]

If one angle of a triangle be greater than another, the side opposite to the greater angle is greater than the side opposite to the smaller angle.

Let ABC be the △.

Given : $\gamma > \beta$.



Required to prove : AB > AC.

Proof.—One of the following statements must be true, and only one can be true :—

either (1) AB = AC
or (2) AB < AC
or (3) AB > AC

If AB, AC were equal, γ and β would be equal (Prop. 11) ; but they are not ; ∴ (1) is not true.

If AB were less than AC, γ would be less than β (Prop. 16) ; but it is not ; ∴ (2) is not true.

∴ (3) must be true, *i.e.* AB > AC.

Exercises.

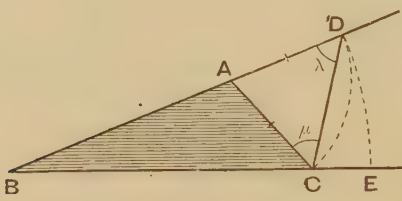
1. If each of the equal sides of an isosceles △ is greater than the third side, prove that the ∠ contained by the equal sides is less than an ∠ of an equilateral △.

2. If a straight line drawn through the vertical angle A of the triangle BAC meet BC in D, and AD be equal to AC, prove that AB is greater than AD.

3. If a point O can be found within a triangle such that OA = AB, prove that AC > AB.

* Prop. 18. [Euclid I. 20.]

Any two sides of a triangle are together greater than the third side.



Let ABC be the △.

Required to prove: AB + AC > BC (1)
also BA + BC > AC (2)
and CA + CB > AB (3)

Proof.—Produce BA to D, making AD = AC.
Then △ DAC is isosceles; ∴ λ = μ
But ∠ BCD > μ; ∴ ∠ BCD > λ

That is, in △ BCD one angle is greater than another;

$$\therefore \left.\begin{array}{l}\text{side BD, which is}\\ \text{opposite to } \angle \text{ BCD}\end{array}\right\} > \left\{\begin{array}{l}\text{side BC, which is}\\ \text{opposite to } \lambda\end{array}\right.$$

Now, BD is made up of BA and AD;
∴ BA + AD > BC

But AD = AC
∴ BA + AC > BC

We may similarly prove (2) and (3).

Note.—Students with a practical turn of mind are often irritated at being asked to give a formal proof of a proposition so obviously true as this is. To appease them we may point out that most people have an exaggerated idea of the distance saved by "cutting off a corner;" the above figure indicates the saving. Thus: describe a circle, centre B, radius BD, cutting BC produced at E; CE is the distance saved.

Exercises.

1. **ABCD** is a quadrilateral, and **AE, CF** are drawn perpendicular to **BD**, meeting it at **E** and **F**; if **AE** = **CF**, prove that **AC** is bisected by **BD.**

2. **ABCD** is a quadrilateral, **AC, BD** its diagonals: if **AC** bisects both the angles **BAD, BCD**, prove that both the triangles **ABD, CBD** are isosceles.

3. In any triangle **ABC**, if **AD** be drawn bisecting the angle **BAC** and meeting **BC** in **D**, and **DE, DF** be drawn parallel to **AC, AB**, meeting them at **E** and **F**, prove that **AEDF** is equilateral.

4. From a point **C** in the straight line **AB** a straight line **CD** is drawn. If **CX** be drawn bisecting $\angle$ **ACD**, and **CY** be drawn equal to **CX** and bisecting $\angle$ **DCB**, and if **XM, YN** are drawn $\perp$ to **AB**, meeting it at **M** and **N**, show that $\triangle$ **XCM** $\equiv$ $\triangle$ **CYN.**

5. The diagonals of a rectangle meet in **O**: prove that of all the straight lines drawn through **O** and terminated both ways by opposite sides, the diagonals are the greatest.

6. **ABC** is an acute-angled $\triangle$ having side **AC** $>$ **AB**; **D** is the mid-point of **BC**: prove that $\triangle$ **ABD** is acute-angled.

7. **ABC** is an equilateral triangle; **D** is taken on **AB**, and **E** on **AB** produced: show that **CD** is less and **CE** is greater than **AB.**

8. Prove that the perpendicular is the shortest straight line which can be drawn from a given point to a given straight line.

9. From a point **P** not on the line **AB** equal lines **PX, PY** are drawn to **AB**: prove that no other line drawn from **P** to **AB** can be equal to **PX.**

10. Prove that any three sides of a quadrilateral are together greater than the fourth side.

11. If a point **P** within a $\triangle$ **ABC** be joined to **A, B, C**, prove that the sum of **PA, PB, PC** is less than the perimeter, but greater than half the perimeter.

12. Prove that the sum of the four sides of a quadrilateral is greater than the sum of the diagonals.

13. In the sides **AB, AC** of a $\triangle$ **ABC** points **D, E** are taken, and **BE, CD** are drawn meeting at **F**:

prove that **DA** + **AE** $>$ **EF** + **FD.**

14. If **P, Q, R** be any three points on the sides **BC, CA, AB** respectively of $\triangle$ **ABC,**

prove that **AP** + **BQ** + **CR** $>$ half the perimeter of the $\triangle$.

* Prop. 19. [Euclid I. 24.]

If two sides of one triangle be equal to two sides of another triangle, but the contained angles be not equal, then the triangle which has the greater contained angle has also the greater base.

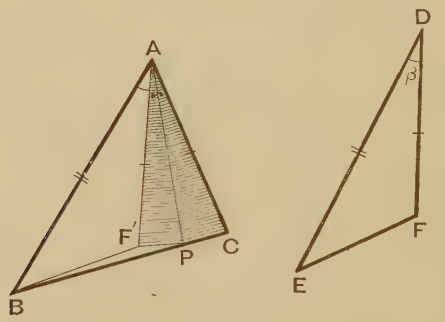


Fig. 1.

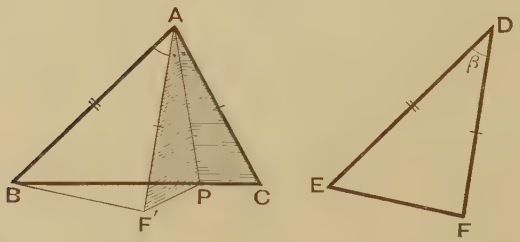


Fig. 2.

Let ABC, DEF be the △s.

Given: AB = DE, AC = DF, ∠ BAC > β } **Required to prove**: BC > EF.

Proof.—Place △ DEF on △ ABC, so that DE lies along AB; DE can be made to fit AB exactly, for they are equal, and both are straight.

Now, $\angle BAC > \beta$ $\therefore$ DF must lie in some position such as AF′ between AB and AC.

F′ may be above BC as in Fig. 1, or below as in Fig. 2 (or on BC, for which no figure is given).

The straight line which bisects $\angle F'AC$ lies between AB and AC, $\therefore$ it cuts BC somewhere; suppose it cuts BC at P.

Join PF′.

$$\triangle PAC \equiv \triangle PAF' \text{ since } \begin{cases} AC = AF' \ (\because AF' \textit{ is } DF) \\ AP = AP \text{ (identical)} \\ \angle PAC = \angle PAF' \end{cases}$$

$$\therefore PC = PF'$$

$$\therefore BP + PF' = BP + PC = BC$$

But BP, PF′ are sides of $\triangle BPF'$; $\therefore BP + PF' > BF'$

$$\therefore BC > BF'$$

i.e. $BC > EF$ ($\because BF'$ *is* EF)

EXERCISES.

1. ABC is a $\triangle$ having $AC > AB$; D is a point in AB produced, and E is a point in AC produced such that $BD = CE$: prove that $BE > CD$.

2. A rhombus is formed by describing two equilateral triangles, one on either side of the same straight line: prove that one diagonal is longer than the other.

3. IMPORTANT EXERCISE.—P is a point within a $\triangle$ ABC; PB, PC are joined.

Prove that $\angle BPC > \angle BAC$.

Also that $AB + AC > PB + PC$.

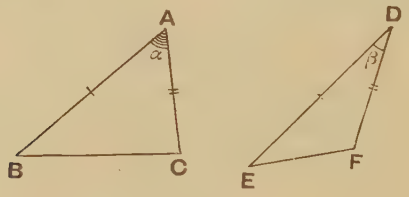
(Produce BP to meet AC at X; use Prop. 5, Cor. 3, and Prop. 18.)

NOTE.—This exercise is the proposition Euclid I. 21.

* Prop. 20. [Euclid I. 25.]

If two sides of one triangle be equal to two sides of another triangle, but the bases be not equal, then the triangle which has the greater base has also the greater angle opposite to it

Let ABC, DEF be the △s.



Given: AB = DE, BC > EF, AC = DF } **Required to prove:** $\alpha > \beta$.

Proof.—One of the following statements must be true, and only one can be true :—

either (1) $\alpha = \beta$
or (2) $\alpha < \beta$
or (3) $\alpha > \beta$

If α and β were equal, BC and EF would be equal.
But they are not; ∴ (1) is not true.

If α were $< \beta$, BC would be < EF.
But it is not; ∴ (2) is not true.

∴ (3) must be true, *i.e.* α is greater than β.

Exercises.

1. XABCY is a polygon, XPQRY is a convex polygon on the same "base" XY, and lying within XABCY.

Prove that (XA + AB + BC + CY) > (XP + PQ + QR + RY).

2. Show that the perimeter of a polygon is always greater than that of a polygon of which the successive corners lie on the sides of the first polygon taken in order.

3. AOB, COD, EOF are diameters of three concentric circles, such that ACE and BDF are two triangles: show that the angle CAE is equal to the angle DBF.

4. Two equal straight lines AB and CD are joined toward opposite parts by the equal straight lines AD and BC, intersecting in O: prove that the triangles OAC and OBD are isosceles.

5. ABC is a △ in which AB = AC; equilateral △s ADB, AEC are described on AB, AC, outside △ ABC; BE and CD intersect at O: prove that OD = OE.

6. Points PQR are taken on the sides BC, CA, AB of an equilateral △ so that BP = CQ = AR; PQ, QR, RP are joined: prove that △ PQR is equilateral.

7. On the sides AB, AC of △ ABC, equilateral △s ABD, ACE are constructed externally; CD and BE are joined: prove that △ BAE ≡ △ DAC.

8. ABC is a right-angled triangle; from the right angle B a line BM is drawn to M, the middle point of AC, and another line BN at right angles to AC: show that the angle MBN is equal to the difference of the angles at A and C.

9. If one side BC of an equilateral triangle ABC be produced to D, so that CD equals BC, show that the angle BAD is a right angle.

10. A, B, C, D, the angular points of a quadrilateral, represent four villages joined by straight roads. Two travellers pass from A to visit all the villages, one going round from A to B, B to C, and C to D; the other from A to C, C to B, and B to D. Prove that the second walks further than the first.

11. The angle contained by one side of a regular polygon and an adjacent side produced is equal to half an angle at the base of an isosceles right-angled triangle: how many sides has the polygon?

12. On the sides BC, CA, AB of an equilateral triangle, the three equilateral triangles BCD, CAE, ABF are drawn: prove that D, E, F are at the corners of an equilateral triangle.

CHAPTER IV.

PARALLELOGRAMS AND AREAS.

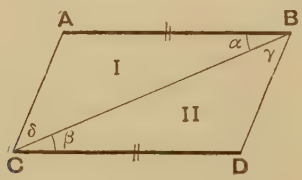
PROP. 21. [Euclid I. 33.]

The straight lines which join the corresponding ends of two equal parallel lines are themselves equal and parallel.

Let **AB, CD** be the equal || straight lines.

Required to prove: **AC** = **BD**, and **AC** || to **BD**.

Join **BC**.



Proof.—**BC** meets || lines **AB, CD**; ∴ α = alternate $\angle$ β.

AB = **CD** (hypothesis)
CB = **CB**
contained $\angle$ α = contained $\angle$ β
} ∴ **I** ≡ **II**

∴ **AC** = **BD**

and δ (opposite to **AB**) = γ (opposite to **CD**)

But γ, δ are alternate $\angle$s made by **BC** with **AC** and **DB**.

∴ **AC, DB** are parallel.

Definition.—A parallelogram is a four-sided figure which has its opposite sides parallel.

Definition.—A rectangle is a parallelogram which has one angle a right angle.

NOTE.—Symbol for parallelogram is $\square$.

The line joining two opposite corners of a four-sided figure is called a **diagonal.**

In the figure of Prop. 21, ACDB is a $\square$. Where there is no risk of confusion, we speak of it as the $\square$ AD (mentioning two opposite corners).

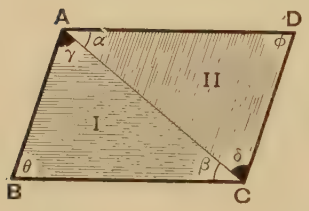
Prop. 22. [Euclid I. 34.]

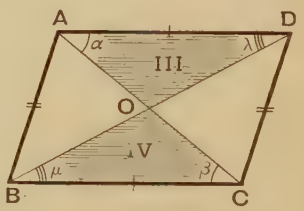
In any parallelogram (1) *opposite angles are equal,* (2) *opposite sides are equal,* (3) *the diagonals bisect the parallelogram,* (4) *they bisect each other.*

Let ABCD be the $\square$, AC a diagonal.

Required to prove:

(1) $\theta = \phi$; (2) AB = CD, BC = AD; (3) I = II.





Proof.—AC cuts || lines AD, BC; $\therefore \alpha =$ alternate $\angle \beta$

AC „ „ AB, CD; $\therefore \gamma =$ „ $\angle \delta$

$\left.\begin{array}{l}\beta = \alpha \\ \gamma = \delta \\ AC = AC\end{array}\right\} \therefore$ I $\equiv$ II *i.e.* AC bisects the $\square$ (3)

Also AB, opposite to β in I, = CD, opposite to α in II (2)

and BC, „ γ „ = AD, „ δ „ (2)

also θ, „ AC „ = ϕ, „ AC „ (1)

Similarly we prove that BD bisects $\square$, and $\angle$ BAD $= \angle$ BCD.

(4) Let diagonals cut at O.

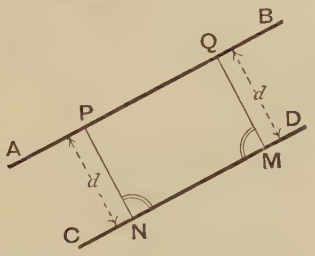
$\left.\begin{array}{l}\lambda = \mu \text{ (alternate } \angle \text{s)} \\ \alpha = \beta \quad \text{„} \\ AD = BC \text{ (proved)}\end{array}\right\} \therefore$ III $\equiv$ IV

$\therefore$ AO, opposite to λ in III, = OC, opposite to μ in IV

DO, „ α „ = OB, „ β „

i.e. O is middle point of AC and of BD.

Corollary 1.—Let AB, CD be || straight lines. In AB take *any* two points P, Q; draw PN, QM ⊥ to CD.



PN, QM are || ∴ PNMQ is a ▱

∴ PN = QM

That is, **"The perpendicular distance *d* between two parallel lines is everywhere the same."**

Corollary 2.—If I be a rectangle having a rt. ∠ α,

then $\alpha + \delta = 2$ rt. ∠s (interior ∠s made by transverse line)

But α is a rt. ∠ ∴ δ is a rt. ∠

also $\gamma = \alpha$, $\beta = \delta$ } (opposite ∠s of ▱)

∴ each angle of a rectangle is a right angle.

Definition.—A square is a quadrilateral having four equal sides and one angle a right angle.

Definition.—A **rhombus** is a quadrilateral having **four equal sides,** but **no** angle a **right angle.**

Exercises.

1. Prove that if the diagonals of a parallelogram are equal, it must be a rectangle.

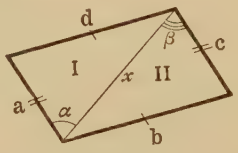
2. If straight lines are drawn bisecting two adjacent angles of the parallelogram, prove that they cut at right angles.

3. Through the corners of a △ lines are drawn ‖ to the opposite sides so as to form another △: prove that this △ is four times as large as the original △.

PROP. 23 (CONVERSE TO PART 2 OF PROP. 22).

If the opposite sides of a quadrilateral be equal, the quadrilateral is a parallelogram.

Let abcd be the quadrilateral.



Given: a = c, b = d } **Required to prove**: abcd is a ▱.

Proof.—Draw the diagonal x.

a = c (hypothesis)
b = d ,,
$x = x$ (identical)
} ∴ I ≡ II

∴ α (opposite to d in I) = β (opposite to b in II)

i.e. the transverse line x makes equal alternate ∠s with a and c;

∴ a and c are ||

Similarly b and d are ||

∴ abcd is a ▱

Corollary 1.—A **square** is a **rectangle** having four equal sides (its opposite sides are equal, ∴ it is a ▱; also one ∠ is a rt. ∠, ∴ it is a rectangle).

NOTE.—The **base** of a ▱ is naturally its lowest side. Often it is convenient to turn a figure round to get a different view of it; this enables us to treat any side we please as base.

The **height** (or **altitude**) of a ▱ is the ⊥ distance between the base and the opposite side. Thus in the second figure of Prop. 24, if we regard AB as the base of ▱ ABCD, k is the height; but if we regard BC as the base, the height is the ⊥ distance between AF and BC.

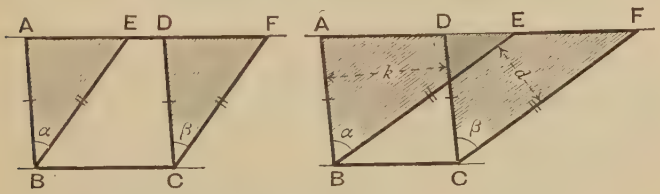
EXERCISE.

If the diagonals of a parallelogram are at right angles, show that the parallelogram must be a rhombus.

PROP. 24. [Euclid I. 35.]

If two parallelograms have the same base and extend to the same straight line parallel to it, the parallelograms are equal in area.

Let ABCD, EBCF be the ▱s having the same base BC, and extending to the same straight line AF



Required to prove: ABCD, EBCF are equal in area.

Proof.—

AB = CD (opposite sides of ▱)
BE = CF „ „
$\alpha = \beta$ (∠s made by pairs of ∥ lines)
$\therefore$ △ ABE ≡ △ DCF

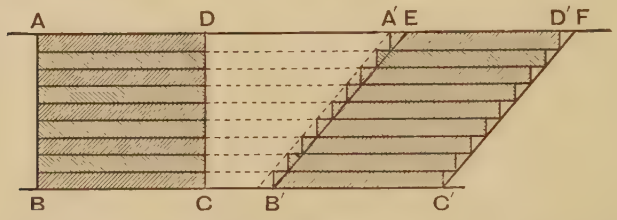
Take △ ABE from figure ABCF, ▱ EBCF remains
„ △ DCF „ „ „ ▱ ABCD „
These remainders are equal;

$$\therefore \text{▱ ABCD} = \text{▱ EBCF}$$

Corollary.—Since a rectangle is a parallelogram, the **area of a parallelogram** is **equal to the area of the rectangle,** which has the **same base** and the **same height.**

At first sight it seems strange that ▱ EBCF in second figure, having such long sides as BE, CF, should be no bigger than ABCD, whose sides AB, CD are so much shorter; but this difficulty disappears when we notice how narrow EBCF is, as judged by the *perpendicular* distance d between its sides BE, CF, in comparison with the perpendicular distance k between AB and CD.

The following is an instructive way of considering this proposition. The rectangle ABCD is cut into equal strips; the lowest is then placed at B′C′, and each of the others pushed to



the right, so as to overhang the one below to the same extent. The zigzag figure thus produced coincides with ▱ EB′C′F, except for the triangular teeth projecting on the left, and indentations on the right; these clearly balance one another.

If the number of strips be increased sufficiently, the rectangle will be like an end view of a pack of cards, and the ▱ will be indistinguishable from the same pack pushed over out of the upright.

Exercises.

1. If a square be described on one side of a rhombus, prove that the area of the square is greater than the area of the rhombus.

2. ABCD, ABFE are two equal parallelograms on opposite sides of AB: prove that CDEF is a parallelogram equal in area to their sum.

3. Any ▱s ABDE, ACFG are described externally on the sides AB, AC of any △ ABC. If DE, FG be produced to meet at L and BM, CN be drawn ∥ and equal to LA, show that—

$$\square BMNC = \square ABDE + \square ACFG$$

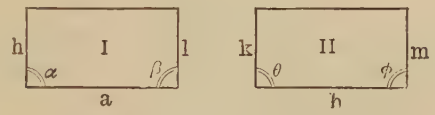
4. ABC is a △ having rt. ∠ at A, and AB = AC. DEF is another △ having rt. ∠ at D, and DE not equal to DF.

If AB + AC = DE + DF, prove that △ DEF < △ ABC.

PROP. 25.

If two rectangles have equal bases and equal heights, they are equal in area.

Let I, II be the rectangles.



Given: base **a** = base **b**, side **h** = side **k** } **Required to prove**: **I** = **II**.

Proof.—Place **I** on **II** so that **a** coincides with **b** (this is possible, for they are equal, and both are straight).

Then **h** lies along **k** ($\because \alpha = \theta$).

Also **h** = **k**, and their lower ends coincide;

$\therefore$ their upper ends coincide.

Similarly **l** coincides with **m**;

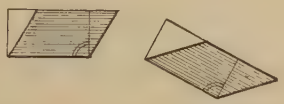
$\therefore$ **I** and **II** coincide entirely;

$\therefore$ **I** $\equiv$ **II**.

Corollary 1 (Converse of Prop.).—If two rectangles be equal in area, and have equal bases, their heights are equal.

For, if we are given base **a** of **I** = base **b** of **II**, then **I** can be placed on **II** so that their bases coincide, and **h** and **l** lie along **k** and **m**; if **h** were not equal to **k**, **I** would either extend beyond **II** or fall short of it, and **I** would not be equal to **II**.

Corollary 2.—If **two parallelograms** have **equal bases** and **equal heights**, they are **equal in area** (I. 36).



For each is equal to the rectangle having the same base

and same height; and these rectangles, having equal bases and equal heights, are equal.

Corollary 3 (Converse of Corollary 2).—If **two parallelograms** having **equal bases** be **equal in area,** their **heights are equal.**

For the rectangles to which they are equal have equal heights.

EXERCISES.

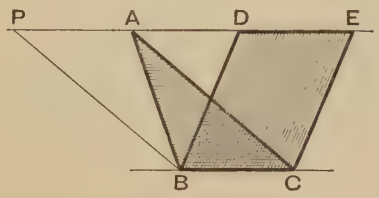
1. **ABCD** is a square; **P** is mid-point of **BC**, **Q** is mid-point of **CD**; **AP, PQ, QA** are joined: prove that $\triangle$ **PQC** = $\frac{1}{8}$ of the square, and $\triangle$ **PQA** = $\frac{3}{8}$ of the square.

2. The base **BC** of a $\triangle$ **ABC** is bisected at **P**. The straight line **AP** is produced to **Q**, so that **PQ** = **AP**, **CQ** is joined: prove (*a*) that $\triangle$ **AQC** = $\triangle$ **ABC** in area; (*b*) that **AP** is less than half the sum of **AB** and **AC**.

PROP. 26. [**Euclid I. 41.**]

If a triangle and a parallelogram have the same base and extend to the same straight line parallel to it, the parallelogram is double the triangle in area.

Let △ ABC and ▱ DBCE have the same base BC, and extend to the straight line AE || to BC.



Required to prove: ▱ DBCE = 2 △ ABC.

Through B draw BP || to CA.

Let it meet EA produced at P.

Proof.—APBC is a ▱, AB is its diagonal;

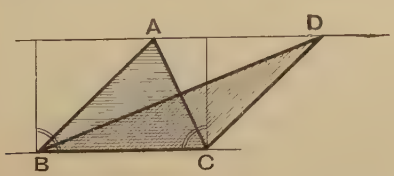
∴ APBC = 2 △ ABC.

Also APBC = DBCE (▱ on same base and same height);

∴ ▱ DBCE = 2 △ ABC.

Corollary 1.—The area of a **triangle** is **half** that of the **rectangle** on the **same base** and of the **same height.**

Corollary 2 [I. 37].—If **two triangles** have the **same base** and extend to the same line parallel to it, they are equal in area. (For each is equal to half of the same rectangle.)

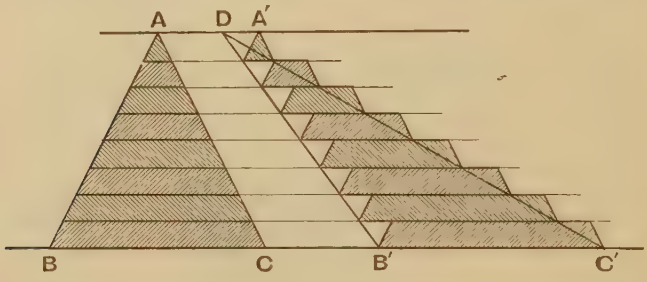


Corollary 3 [I. 38].—If **two triangles** have **equal**

bases and **equal heights**, they are equal in area. (For they are halves of equal rectangles.)

Corollary 4 [I. 39 and I. 40].—Converse of Corollary 3.

The figure indicates another proof of Corollaries 3 and 4. ABC is a △ cut into strips. The lowest strip being placed at



B′C′, each of the others is pushed to the left; the zigzag figure produced coincides with △ DB′C′, except for the indentations on the left and the projecting teeth on the right. By increasing the number of strips very much we can make the strips extremely narrow, and then the effect of the teeth and indentations is negligible; hence we see that △ ABC = △ DB′C′ (in area).

[The projecting teeth and the indentations are, in fact, equal here, just as they were for the ▱s; but even if they had not been equal, the proposition would have been true, since in the long run both teeth and indentations become negligible compared with the size of the strips.]

Exercises.

1. The sides BC, CA of a triangle ABC are produced to D and E respectively, so that CD is equal to BC and AE is equal to AC: prove that the triangles EAD and ABC are equal.

2. ABP and AQC are two straight lines, and the triangles APQ and ABC are equal; BQ, PC are joined: prove that BQ is ∥ to PC.

Measurement of Areas.

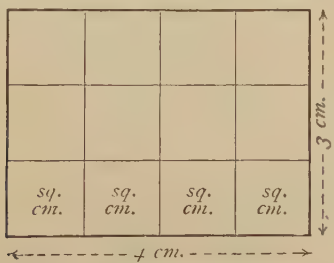
The area of a figure is measured by finding out how many times it contains some standard or "unit" area; for large areas, the acre, the square mile, or the square kilometre are convenient units; for our purpose, the square inch, square centimetre, or square millimetre are better.

A square centimetre is a square having each side 1 cm. long; it can be cut, as in figure, by horizontal lines into ten strips, each 1 mm. deep and 10 mm. long. Each strip can be cut by vertical lines into ten small squares, each of which is a square millimetre.

Hence 1 sq. cm. = 100 sq. mm.

Notice, also, that 1 sq. inch can be divided into 100 square "tenths of an inch."

So a rectangle whose sides each contain an exact number



of centimetres, say 3 cms. and 4 cms., can be cut into horizontal strips, each of which can be cut into 4 sq. cms.

Hence number of sq. cms. = 3×4

i.e. area of rectangle = $3 \times 4 = 12$ sq. cms.

If the sides of the rectangle do not contain exact numbers of centimetres, suppose next they contain exact numbers of millimetres; say the sides are 2·8 cms. and 3·3 cms., *i.e.* 28 mm. and 33 mm.

Then dividing into 28 strips, each 1 mm. wide, we have—

$$\text{area of rectangle} = 28 \times 33 \text{ sq. mm.} = \frac{28 \times 33}{100} \text{ sq. cms.}$$

$$= \frac{28}{10} \times \frac{33}{10} \text{ sq. cms.} = 2{\cdot}8 \times 3{\cdot}3 \text{ sq. cms.}$$

$$= 9{\cdot}24 \text{ sq. cms.}$$

If the sides do not contain exact numbers of millimetres, more accurate measurements must be made. Now in all measurement, however refined, there is some limit to the accuracy attainable; suppose, for example, the limit to our accuracy is ·001 cm.; we can in imagination cut a square centimetre into 1000 strips, each strip containing 1000 minute squares; then each of these minute squares is a "square-thousandth of a centimetre."

Hence 1 sq. cm. = 1 million "square-thousandths."

Now let the sides of the rectangle be 2·784 cms. and 3·427 cms.: the rectangle can be cut into 2784 strips each containing 3427 of the minute squares;

∴ area of rectangle = 2784 × 3427 millionths of sq. cms.

$$\begin{array}{r} 2{\cdot}784 \\ 3{\cdot}427 \\ \hline 8352 \\ 11136 \\ 5568 \\ 19488 \\ \hline 9{\cdot}540768 \end{array}$$

$$= \frac{2784 \times 3427}{1000000} = \frac{2784 \times 3427}{1000 \times 1000}$$

$$= 2{\cdot}784 \times 3{\cdot}427 \text{ sq. cms.}$$

$$= 9{\cdot}540768 \text{ sq. cms.}$$

Similarly in all cases and for any units—

area of rectangle = length × breadth

When sides of the rectangle are 2·7839 cms. and 3·427 cms., the area is 9·5404253 sq. cms.

$$\begin{array}{r} 2{\cdot}7839 \\ 3{\cdot}427 \\ \hline 83517 \\ 111356 \\ 55678 \\ 194873 \\ \hline 9{\cdot}5404253 \end{array}$$

Notice that this area differs from the former area in the "fourth decimal figure," which is the **"fifth significant figure."** This change in area arises from a change from 2·784 cms. to 2·7839 cms. in the length of a side; by supposition, such a change cannot be detected by our measurement. Hence the true area of the rectangle, whose sides *are* found to be 2·784 cms.

and 3·427 cms., is just as likely to be 9·5404253 sq. mm. as to be 9·5402768 sq. cms., and it is just as likely to be a little greater or a little less. The **only reliable part** of the result is the **first four figures**; the fifth figure is unreliable, the rest worthless.

```
2·7'8'4
3·427
------
 8352
 11136
   556
   196
------
9·5408
```

Hence the multiplication should be worked by approximation methods, keeping the fifth significant figure to protect the fourth in "carrying;" but the fifth figure should, as a rule, be rejected in quoting the result.

Area = 9·541 sq. cms.

NOTE.—In more refined measurements, such as those made in the Ordnance Survey, five-figure and even six-figure accuracy is attained; then calculations may be carried to six or seven figures.

In schoolwork three-figure accuracy is possible.

Area of a Square.—A square is a rectangle in which **length = breadth;**

$\therefore$ area of square = length of side $\times$ length of side
= $(\text{side})^2$

e.g. if side of square = 7 inches, area = 7^2 = 49 sq. ins.
" " = 13 metres, area = 13^2 = 169 sq. metres

Other parallelograms and triangles cannot, like rectangles, be cut into squares, but Props. 24, 25, 26 enable us to find their areas. Thus—

Area of $\square$ = area of rectangle of same height and base
= height of $\square$ $\times$ base

Area of $\triangle$ = $\frac{1}{2}$ area of rectangle of same height and base
= $\frac{1}{2}$ height of $\triangle$ $\times$ base

EXAMPLE.—Height of $\square$ = 10·56 metres; base = 33·42 metres.

```
10·5'6
33·42
------
 3168
 3168
  424
   20
------
352·92
```

Area = 10·56 × 33·42
= 352·9 sq. metres

EXAMPLE.—Height of $\triangle$ = 76·53 ins. ; base = 45·79 ins.

```
  76·53
  45·79
 ------
 30612
  38265
   5355
    693
 ------
2)3504·33
  1752·2
```

$$\text{Area} = \tfrac{1}{2} \times 76{\cdot}53 \times 45{\cdot}79$$
$$= 1752{\cdot}2 \text{ sq. ins.}$$

NOTE.—Here we keep five figures in result ; for 1752·2, which exceeds 999·9 by so little, practically corresponds to "four-figure" accuracy.

EXAMPLE.—Area of $\triangle$ = 5·623 acres ; height = 24·58 chains : find base.

```
2·4·5·8)11246(4574
       1414
        184
          9
```

$$\text{base} = \frac{2 \text{ area}}{\text{height}} = \frac{112{\cdot}46}{24{\cdot}58} = 4{\cdot}574 \text{ chains}$$
$$= 4 \text{ chains } 57 \text{ links}$$

[N.B.—1 acre = 10 sq. chains.]

EXERCISES.

Find following areas :—

1. Square ; side = 23 cms.
2. Rectangle ; length = 12 cms. ; breadth = 9 cms.
3. „ „ = 42·3 cms. ; „ = 33·8 cms.
4. „ „ = 6·72 chains ; „ = 4·98 chains.
5. $\square$; base = 12·5 ins. ; height = 8·23 ins.
6. $\triangle$ „ = 5·92 metres ; height = 4·17 metres.

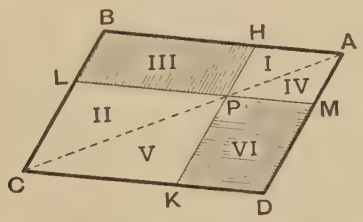
Find bases of the following :—

7. $\square$; area = 146 sq. cms. ; height = 6·6 cms.
8. $\triangle$ „ = 587 sq. ins. ; „ = 18·5 ins.

[If through any point P on a diagonal of a $\square$ ABCD two lines be drawn $\parallel$ to the sides, four small $\square$s are formed ; two of these have diagonals AP, CP, which are parts of the diagonal AC of the original $\square$, they are called **parallelograms about the diagonal** (an awkward expression which is not often needed) ; the $\square$s III and VI, which complete the figure, are called **complements** of the other two.]

PROP. 27. [Euclid I. 43.]

Complements of parallelograms about a diagonal of a parallelogram are equal.



Let III and VI be complements of $\square$s about the diagonal APC of $\square$ ABCD.

Required to prove: III = VI.

Proof.—I + II + III = IV + V + VI ($\because$ AC bisects ABCD).

But $\begin{cases} \text{I} = \text{IV} & (\because \text{AP bisects HM}) \\ \text{II} = \text{V} & (\because \text{CP} \quad ,, \quad \text{LK}) \end{cases}$

$\therefore$ III = VI (by subtraction)

EXERCISES.

1. ABCD is a $\square$, and a straight line drawn parallel to AB meets AD at P, AC at Q, and BC at R: prove that—

$$\triangle \text{APR} = \triangle \text{AQD}$$

2. HF and EG are the parallelograms about the diagonal AC of a $\square$ ABCD: prove that—

$$\triangle \text{AEG} + \triangle \text{AHF} = \triangle \text{CHF} + \triangle \text{CEG}$$

3. (*a*) Prove that every straight line which bisects the area of a $\square$ must pass through the point of intersection of its diagonals.

(*b*) State and prove the converse of (*a*).

4. $\triangle$s ABC and DBC are on same base BC and are equal. If ABCK is a $\square$, and AC and BD meet in O, prove that—

$$\triangle \text{BOC} - \triangle \text{AOD} = \triangle \text{DCK}$$

Exercises.

1. ABCD is a $\square$; P, Q, R, S are points in AB, BC, CD, DA, such that AP = CR and BQ = DS: prove that PQRS is a $\square$.

2. ABC is a $\triangle$; from a point P in BC, lines PM, PN are drawn $\parallel$ to CA, BA, meeting AB, AC in M, N; PN is produced to Q so that PQ = AB: if the $\square$ CNQL be completed, prove that it is equal to AMPN.

3. The diagonals AC, BD of a quadrilateral ABCD intersect at O, and the $\square$s OAEB, OBFC, OCGD, ODHA are completed: prove that EFGH is a $\square$ and is = twice quadrilateral ABCD.

4. ABCD is a $\square$; P is the mid-point of BC, and AP and DC are produced to meet in N: prove that $\triangle$ DPN is half $\square$ ABCD.

5. ABCQP is a convex polygon such that the sides AP and BC are $\parallel$, and $\triangle$ ABC = $\triangle$ BPQ in area: prove that the straight lines BP and CQ are $\parallel$.

6. If a quadrilateral is bisected by each of its diagonals, it is a parallelogram.

7. P is any point inside a $\square$ ABCD: prove that $\triangle$ PAB + $\triangle$ PCD is equal to half the $\square$.

8. The sides of a $\triangle$ ABC are produced towards the same parts so that BC = CD, CA = AE, and AB = BF: find how many times the triangle ABC is contained in the triangle DEF.

9. Through A, B, C are drawn three parallel straight lines to meet the opposite sides of $\triangle$ ABC (produced if necessary) at A′, B′, C′: prove that $\triangle$ A′B′C′ = twice $\triangle$ ABC.

10. ABCD is a $\square$; any point P on the diagonal AC is joined to B and D: prove that $\triangle$ APB = $\triangle$ APD.

11. AB and ECD are two $\parallel$ straight lines; BF, DF are drawn $\parallel$ to AD, AE respectively: prove that $\triangle$ ABC = $\triangle$ DEF.

12. AEKG, KHCF are parallelograms about the diagonal AC of a parallelogram ABCD: show that EG is parallel to HF.

13. ABCD is a parallelogram; E and F are the middle points of AB and CD; P and Q are any points in AD and BC respectively: prove that EF bisects PQ.

14. ABCD is a $\square$; AD is bisected at E, and CE is joined; BA and CE are produced to meet at F: prove that the area of $\triangle$ FBC is equal to that of the $\square$.

15. ABCD is a $\square$, and the straight line XY is drawn parallel to the diagonal BD, meeting the side BC at X and the side CD at Y: prove that $\triangle$ ABX is equal to $\triangle$ ADY.

CHAPTER V.

SIMPLE CONSTRUCTIONS.

In the figures of this chapter note that as a rule—

(1) **Given lines** are drawn **thick.**

(2) **Lines required** to be constructed, **less thick.**

(3) **Lines helping** in the construction, **fine.**

(4) **Lines** needed only for the **proof**, or to make the figure easier to understand, are "**broken.**"

These rules are occasionally broken where parts of the figures have been shaded to bring them into prominence.

In describing circles, the complete circle is not drawn when only a part is required.

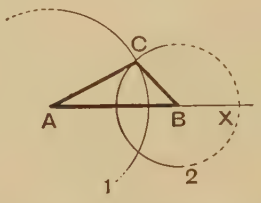
When any part of a figure is drawn by the help of a previous construction, the necessary construction lines are shown; but they are not lettered, and no reference is made to them in the text.

Note.—Pencils used in constructing figures should be sharpened by giving them a sharp flat edge like the edge of a chisel; a rounded point wears away too quickly.

Construction 1. [Euclid I. 22.]

Required to construct a triangle, being given the lengths of the sides.

For example, let the given lengths be 17 mm., 13 mm., 8 mm.
Draw a straight line **AX**.
Mark off along it **AB** = 17 mm.



Describe circle **1**, centre **A**, radius 13 mm.
„ „ **2**, „ **B**, „ 8 mm.
These circles cut each other at two points.
Denote one of these by **C**. Join **AC, BC**.

Every radius of **1** = 13 mm. ∴ the particular radius **AC** = 13 mm.
„ „ **2** = 8 mm. ∴ „ „ „ **BC** = 8 mm.

∴ **ABC** has its sides of the given length.

Note.—If the three given lengths be equal, the triangle described is equilateral.

Exercises.

Construct △s with the following dimensions, where a, b, c are the sides **BC, CA, AB**. [In two cases the construction fails; notice carefully **how** and **why** it fails.]

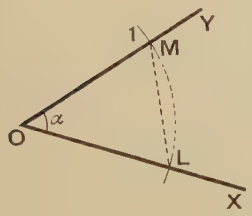
1. $c = 2''$, $a = 1''$, $b = 1\frac{1}{2}''$.
2. $c = 3''$, $a = 2{\cdot}2''$, $b = 2{\cdot}8''$.
3. $c = 5$ cms., $a = 3{\cdot}8$ cms., $b = 6{\cdot}2$ cms.
4. $c = 2{\cdot}8$ cms., $a = 4{\cdot}9$ cms., $b = 4{\cdot}4$ cms.
5. $c = 3''$, $a = 1{\cdot}5''$, $b = 1{\cdot}25''$.
6. $c = 1{\cdot}2''$, $a = 0{\cdot}8''$, $b = 2{\cdot}5''$.

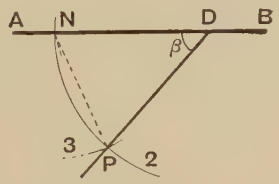
Keep these △s for use after Constructions 2 and 6.

CONSTRUCTION 2. [Euclid I. 23.]

Required to draw through a given point in a straight line a line making with it an angle equal to a given angle.

Let D be the given point in AB.
„ α „ „ angle XOY





Describe ⊙ 1, centre O, any radius.
„ 2, „ D, same radius.
Let 1 cut OX at L, OY at M.
„ 2 „ AD at N.
Describe ⊙ 3, centre N, radius = ML.
Let 3 cut 2 at P.
Join PD; then β is required ∠.

Proof.—

$$\triangle OML \equiv \triangle DNP, \text{ since } \begin{cases} OM = DN \text{ (radii of equal } \odot\text{s)} \\ OL = DP \quad \text{„} \quad \text{„} \\ ML = NP \text{ (NP is radius of 3)} \end{cases}$$

$\therefore$ β opposite to NP = α opposite to ML

Use of Protractor.

To **make a required angle**, say 48°, **with a protractor** (which may in shape be either a semicircle or a rectangle). Rule a line such as **OB** in figure, p. 85. Place the protractor so that the 48° graduation on the upper edge comes on **OB**, taking care that the mark which indicates the middle point of the edge is exactly at **O**. Then rule **OA** along the lower edge of the protractor.

To **measure an angle**, say ϕ in the figure, p. 85, **with a protractor.** Place lower edge of protractor along **OP** with middle point at **O**; then read the number of degrees marked on the upper edge of protractor at the point where **OB** emerges from below it. If the figure be small, it may be necessary to produce **OB** before the $\angle$ can be measured.

Exercises.

1. Measure the $\angle$s of each $\triangle$ constructed in last set of exercises.

2. Construct a rhombus, each side = 2″, one $\angle$ = 55°.

3. Construct a rhombus, each side = 5·8 cms., one $\angle$ = given $\angle$ a.

4. Construct a rhombus, each side = 2·6″, one diagonal = 4″: measure the other diagonal; also calculate its length, using Prop. 35; measure the $\angle$s.

5. Construct a $\square$, sides 6 cms., 2·8 cms., one diagonal = 7 cms.; measure the $\angle$s and the other diagonal.

6. Construct a quadrilateral **ABCD**, given sides **AB** = 2 cms., **BC** = 2·5 cms., **CD** = 4 cms., **DA** = 2·9 cms., diagonal **AC** = 4·2 cms.; measure $\angle$s and the other diagonal. [First construct $\triangle$ **ABC** of which all sides are given; then describe $\odot$ centre **A**, radius 2·9 cms., and $\odot$ centre **C**, radius 4 cms.]

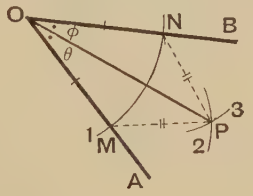
7. Construct a pentagon **ABCDE**, given **AB** = 2″, **BC** = 1·5″, **CD** = 2·25″, **DE** = 1·25″, **EA** = 1·75″; given also the measurements **AC** = 3″, **AD** = 2·5″.

8. Describe an isosceles $\triangle$ whose vertical $\angle$ is four times each of the $\angle$s at the base.

Construction 3. [**Euclid I. 9.**]

Required to bisect a given angle.

Let AOB be the given angle.
Describe a ⊙ 1, centre O, any convenient radius.



Let 1 cut OA and OB at M and N.
Describe ⊙ 2, centre M, any convenient radius.
„ 3, „ N, same radius.
These ⊙s cut at two points. Let P be the one furthest from O.
Join OP; it is the required bisector.

Proof.—Join PM, PN.

OP = OP
OM = ON (radii of same ⊙)
PM = PN („ equal ⊙s)
} ∴ △ POM ≡ △ PON

∴ θ opposite to PM = φ opposite to PN

Exercises.

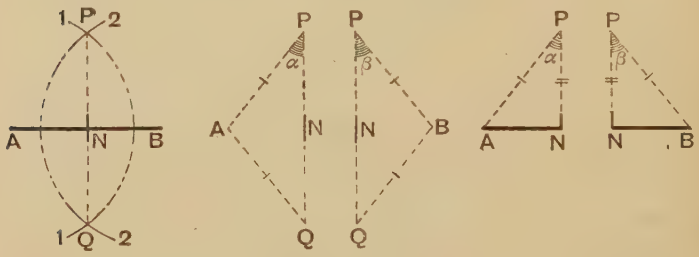
1. Construct an equilateral △, each side 28 mm.; bisect an ∠; measure each half with a protractor.

2. Draw with protractor an ∠ 78°; bisect by construction; check accuracy by measuring each half.

Construction 4. [Euclid I. 10.]

Required to bisect a given straight line.

Let AB be the given line.
Describe ⊙ 1, centre A, any convenient radius.
„ 2, „ B, same radius.



They cut at two points, P and Q.
Join PQ, cutting AB at N.
N is the required middle point of AB.

Note.—To avoid confusion, the △s required in proof are drawn detached from the constructed figure.

Proof.—Join AP, BP, AQ, BQ.

$$\left.\begin{array}{l} AP = BP \text{ (radii of equal circles)} \\ AQ = BQ \quad „ \quad „ \quad „ \\ PQ = PQ \text{ (identical)} \end{array}\right\} \therefore \triangle APQ \equiv \triangle BPQ$$

$\therefore \alpha$ opposite to AQ $= \beta$ opposite to BQ

$$\left.\begin{array}{l} AP = BP \\ PN = PN \\ \alpha = \beta \end{array}\right\} \therefore \triangle APN \equiv \triangle BNP$$

$\therefore$ AN = BN

Note.—With a good pair of "dividers," a line may be bisected by trial almost as quickly and accurately as by construction.

Exercise.

Draw lines $4\frac{1}{4}$", 9 cms., and 15 mm. long; bisect each by construction; check your work by measurement.

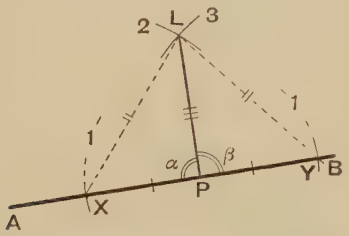
Construction 5. [Euclid I. 11.]

Required to draw a straight line at right angles to a given line from a given point in it.

Let AB be the given straight line.

„ P „ „ point in it.

First Method.—Describe ⊙ 1, centre P, any convenient radius. Let it cut AB at X and Y.



Describe ⊙ 2, centre X, any convenient radius.

„ 3, „ Y, same radius.

These ⊙s cut at two points; let L be one of them.

Join PL; it is the required line at right ∠s to AB.

Proof.—

PX = PY (radii of same circle)
XL = YL („ equal circles)
PL = PL
} ∴ △ PXL ≡ △ PYL

∴ $\alpha = \beta$, *i.e.* each is a rt. ∠

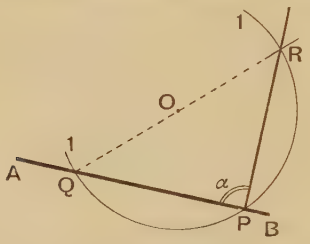
Note.—To get a good result, choose radius PX as long as possible, and radius XL rather less than three-fourths of XY. The student should experiment on this construction, trying different values for XL from about half XY up to great values.

Second Method.—To be used when P is near one end of AB.

Take any point O not in AB, and well to the left of P.

Describe ⊙ 1, centre O, radius OP.

1 cuts AB at P and at another point. Call this point Q.



Through Q draw the diameter QOR.
Join PR; it is the required ⊥ to AB.

Proof.—Draw the radius OP, then the △s OPQ, OPR are isosceles; the proof, which will be given in full in Prop. 45, is an exercise on Prop. 10 and Prop. 5.

Exercises.

1. Construct an isosceles △, given one of the base ∠s; length of equal sides each 48 mm.

2. Construct an isosceles △ with given base, and of height 1·8″. [Draw a line as PQ in Const. 4, bisecting base and ⊥ to it; mark off 1·8″ along the ⊥; join to ends of base.]

3. Construct an isosceles △ with given vertical ∠, and base = $2\frac{1}{2}$″. [Construct OX bisecting given ∠. Through any point P in OX draw a ⊥ to it; cut off $1\frac{1}{4}$″ each side of P. Draw lines ∥ to OX; they meet arms of given ∠ at corners of required △.]

4. Construct a right-angled △; hypotenuse = 2·7″, one side = 2·2″. [Draw line AB 2·2″ long; describe ⊙ centre A, radius 2·7″; at B erect a ⊥ cutting ⊙ at P. PAB is required △.]

5. Construct a right-angled △, with hypotenuse = 5 cms. and one acute ∠ equal to a given ∠. [Draw AB = 5 cms.; find middle point O. Describe ⊙ centre O, radius OA. At A construct ∠ = given ∠; let its arm cut ⊙ at P. PAB is required △.]

6. Construct a △, height = $1\frac{3}{4}$″, sides = $2\frac{1}{4}$″ and $2\frac{3}{4}$″; measure the base and calculate the area. [Draw AN = $1\frac{3}{4}$″; construct XNY ⊥ to AN; describe ⊙s, centre A, radius $2\frac{1}{4}$″, $2\frac{3}{4}$″.]

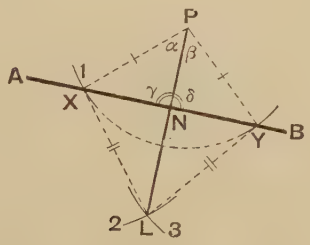
7. Construct a △, height = 6·7 cms., base ∠s = given ∠s α, β. [Draw AN = 6·7 cms.; construct PAQ, XNY, each ⊥ to AN; at A make ∠ PAB = α, and ∠ QAC = β. AB and AC mark off the base of required △ on XY.]

Construction 6. [Euclid I. 12.]

Required to draw a straight line perpendicular to a given straight line, from a given point not on it.

Let AB be the given straight line.
„ P „ „ point not on it.

First Method.—Describe ⊙ 1, centre P any convenient radius. Let it cut AB at X and Y.



Describe ⊙ 2, centre X, any convenient radius.
„ 3, „ Y, same radius.
Let L be the point, furthest from P, where they cut.
Join PL, cutting AB at N; it is the required line ⊥ to AB.

Proof.—

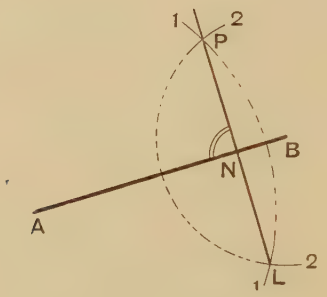
PX = PY (radii of same circle)
XL = YL („ equal circles)
PL = PL
} ∴ △ PXL ≡ △ PYL

∴ $\alpha = \beta$

Now, PX = PY
PN = PN
$\alpha = \beta$
} ∴ △ PXN ≡ △ PYN

∴ $\gamma = \delta$, *i.e.* each is a right ∠

Second Method.—To be used when P is nearly opposite one end of AB.



Describe ⊙ 1, centre A, radius AP.
„ 2, „ B, „ BP.
⊙ 1 cuts ⊙ 2 at another point. Call this point L.
Join PL, cutting AB at N; it is the required line ⊥ to AB.

Proof.—The proof is similar to that for first method, and is left as an exercise.

NOTE.—If P be given in a position much further to the right, this construction gives a line PL ⊥ to AB produced.

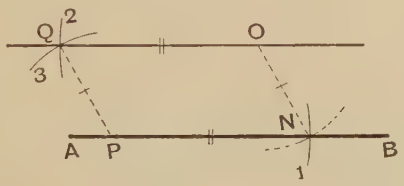
NOTE ON CONSTRUCTIONS 5 AND 6.—With a good set-square and a good straight-edge, perpendiculars can be drawn accurately without any construction, hence draughtsmen rarely use constructions; but a schoolboy using a common set-square will get greater accuracy by construction.

EXERCISE.

In the △s constructed in Exercises, p. 82, draw perpendiculars from C to AB; measure them; calculate area of △s.

Construction 7. [Euclid I. 31.]

Required to draw through a given point a straight line parallel to a given straight line.



First Method (with compasses and straight-edge).

Let AB be the given straight line;

O „ „ point.

In AB take any point P near A.

Describe ⊙ 1, centre P, any radius.

„ 2, „ O, same radius.

„ 3, „ P, radius = ON.

Let 3 cut 2 at Q.

Join OQ; it is the required line parallel to AB.

Proof.—In the quadrilateral OQPN—

OQ = opposite side PN (radii of equal circles)

PQ = „ „ ON „ „ „

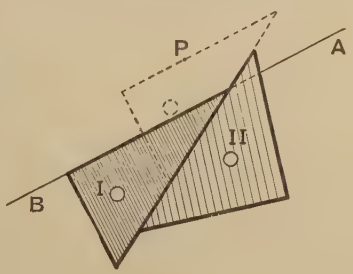
∴ OQ is ‖ to AB (Prop. 23)

Second Method (with two set-squares).

Place the set-square I with an edge along the given line AB.

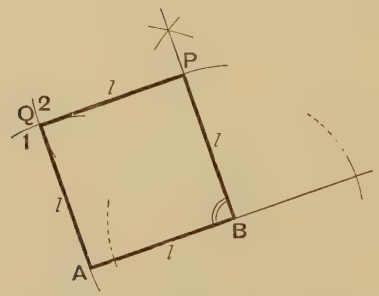
Then place set-square II along the hypotenuse of I.

Hold II firm, and slide I along it till its edge passes through the given point P. A line drawn through P along the edge of I is clearly ‖ to AB, for in each position the edge of I makes the same angle with the hypotenuse of II.



Construction 8. [Euclid I. 46.]

Required to construct a square on a given straight line.



Let AB be the given line, of length *l*.

At B draw BP $\perp$ to AB, making BP = *l*.
Describe $\odot$ 1, centre A, radius *l*.
„ 2, „ P, „ *l*.
These $\odot$s cut at B, and also at another point Q.
Join PQ, AQ; ABPQ is the required square.

(For $\angle$ at B is a rt. $\angle$, and each side is of the same length *l*.)

Construction 9.

Required to divide a given straight line into a given number of equal parts.

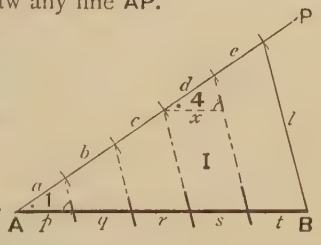
For example, let it be required to divide AB into five equal parts.

First Method.—Through A draw any line AP.

Mark off *a*, *b*, *c*, *d*, *e* along AP, all equal and of any convenient length.

Draw *l*, joining B to the further end of *e*.

Through end points of *a*, *b*, *c*, *d* draw, with help of set-square, lines ‖ to *l*, dividing AB into parts *p*, *q*, *r*, *s*, *t*; these are the required equal parts.



Proof.—Through lower end of *d* draw *x* ‖ to AB.

Two ∠s of 1 = two corresponding ∠s of 4
side *a* = side *d*
} ∴ △ 1 ≡ △ 4

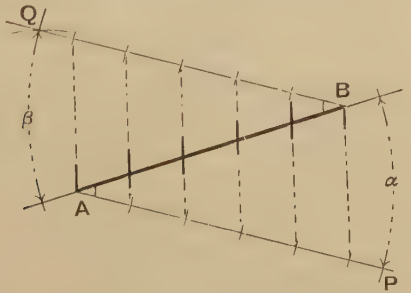
∴ *p* = *x*

But *s* = *x* (∵ I is ▱)

∴ *p* = *s*, and similarly *p* = *q* = *r* = *t*

Second Method.—Through A draw any line AP.

Through B draw BQ, making $\beta = \alpha$.



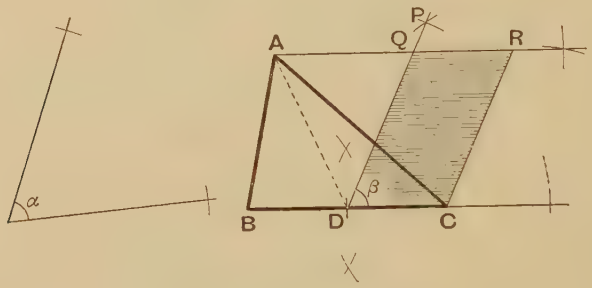
Mark off five equal lengths along each of AP and BQ.

Join as in the figure; obviously AB is divided as required.

Formal proof is left as an exercise.

Construction 10. [Euclid I. 42.]

Required to construct a parallelogram equal to a given triangle and having an angle equal to a given angle.



Let ABC be the given △.

α „ „ ∠.

Bisect BC at D.

Draw DP, making $\beta = \alpha$.

„ AQR || to BC, cutting DP at Q.

Prick off QR = DC; join CR.

QDCR is the required ▱.

Proof.—If AD be drawn—

△ BDA = △ CDA (∵ BD = CD, and △s are of same height)

∴ △ ABC = 2 △ CDA

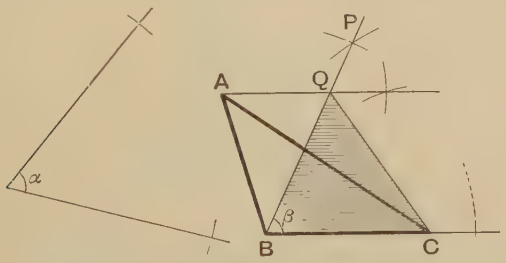
▱ QC = 2 △ CDA (∵ on same base and of same height)

∴ ▱ QC = △ ABC

Also $\beta = \alpha$ (construction)

CONSTRUCTION II.

Required to construct a triangle equal to a given triangle and having an angle equal to a given angle.



Let ABC be the given △.

α „ „ ∠.

Draw BP, making $\beta = \alpha$.

„ AQ || to BC, cutting BP at Q.

Join CQ.

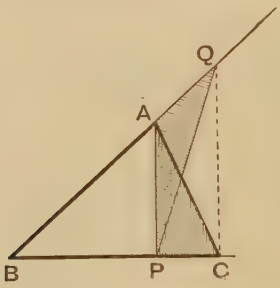
QBC is the required △.

Proof.—△ QBC = △ ABC (∵ on same base and of equal height)

Also $\beta = \alpha$ (construction)

Construction 12.

Required to construct a triangle equal in area to a given triangle, having one side of given length, and one angle unchanged.



Let ABC be the given △, B the ∠ to be left unchanged.

From BC (produced if necessary) cut off BP equal to the given length.

Join AP.

Through C draw CQ || to PA, meeting BA produced at Q.

Join PQ.

BPQ is required △.

Proof.—△ APQ = △ APC (∵ on same base AP and of same height);

∴ △ APQ + △ APB = △ APC + △ APB.

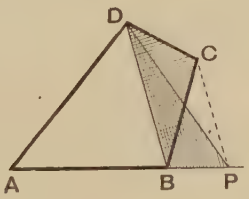
i.e. △ BPQ = △ ABC

Note 1.—To construct a △ having one side of given length, one ∠ equal to a given ∠, and also its area equal to a given △. (1) By Const. 12 make a △ of the required area having its side of the given length; (2) by Const. 11 make on the same base an equal △ having the required ∠.

Note 2.—The methods of Consts. 11 and 12 may be used to construct ▱s under similar conditions. If this be found difficult, bisect the given ▱ by drawing a diagonal, then construct a △ equal to the half ▱, and finally construct the ▱ of which the new △ is one half.

CONSTRUCTION 13.

Required to construct a triangle equal in area to a given quadrilateral.



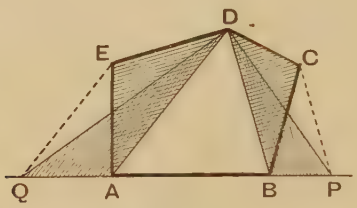
Let ABCD be the given quadrilateral.
Join BD.
Through C draw CP || to DB and meeting AB produced at P.
Join PD.

ADP is the required △.

Proof.—△ BDP = △ BDC (∵ on same base BD and of same height);
∴ △ BDP + △ ABD = △ BDC + △ ABD
i.e. △ ADP = ABCD

Construction 14.

Required to construct a triangle equal in area to a given pentagon.



Let ABCDE be the pentagon.
Join AD, BD.
Through C draw CP || to DB and meeting AB produced at P.
„ E „ EQ „ DA „ BA „ Q.
Join PD, QD.

PDQ is the required △.

Proof.—△ BDP = △ BDC (∵ on same base BD and of same height)
△ ADQ = △ ADE (∵ on same base AD and of same height)
∴ △ BDP + △ ADQ + △ ABD = △ BDC + △ ADE + △ ABD
i.e. △ PDQ = ABCDE

Note.—The method of Consts. 13 and 14 repeatedly applied enables us to reduce any polygon to a △ of equal area.

Exercises.

1. Construct a quadrilateral, AB = 2″, BC = 2·4″, CD = 1·8″, DA = 1·5″; diagonal AC = 2·8″: reduce to an equal △; from it obtain an equal rectangle.

2. Construct a pentagon, AB = 3·3 cms., BC = 3 cms., CD = 2·2 cms., DE = 4 cms., EA = 4 cms.; also given AC = 4·5 cms., AD = 5·5 cms.: reduce to an equal △, and calculate the area.

3. Construct a quadrilateral ABCD, ∠ at A = 50°, ∠ at B = 110°; AB = 2″, AD = $1\frac{3}{4}$″, BC = $1\frac{1}{4}$″: measure the other ∠s; reduce the quadrilateral to an equal △ and calculate area.

4. Divide a given ∠ into two parts so that one part shall be three times as large as the other.

5. Describe an equilateral △ whose perimeter is equal to the perimeter of a given △.

6. Show how to describe an isosceles △ having the vertical ∠ treble of each of the ∠s at the base.

7. On a given base describe a quadrilateral such that each of its three sides is half the base.

8. Construct a △ with ∠s α, β, γ such that $\alpha = \frac{1}{2}(\beta + \gamma)$, and one side containing α = twice the other side.

9. Describe a right-angled △ equal in area to a given △, and having one of the sides containing the rt. ∠ equal to a given straight line.

10. Through a given point draw a straight line such that the part of it intercepted between two given ‖ straight lines may be equal to a given line.

Can this problem always be solved?

11. On a given straight line describe a △ having one of its angles at the base five times and the other six times as great as the angle at the vertex.

12. On the longest side of a given scalene △ describe a rhombus equal to the △.

13. Draw a line 6·3 cms. long: divide it into seven equal parts.

14. Draw a line 3·3″ long: divide it, by construction, into three equal parts, and check by measurement.

15. Construct an isosceles △, base = 23 mm., sides = 34 mm.

16. **A, B** are two given points: find two other points, **P, Q**, whose distances from **A, B** are equal to **AB**. Show that **PQ** when joined is perpendicular to and bisects **AB**.

17. Draw a line parallel to a given line at a distance 4 cms. from it. [Take two points in given line near the ends; erect perpendiculars; cut off 4 cms. along each, and join.] **Or** [Describe two ⊙s, each 4 cms. radius, with centres on given line; place a straight-edge so as to touch both ⊙s, and rule the required line].

18. Draw through a given point a line meeting a given line at an angle equal to a given angle. [Take any point on given line; by Const. 2 make ∠ = given ∠; through given point draw a ‖ line by Const. 7.]

19. Given two lines **OX, OY** radiating from **O**, draw through a given point **A** a line making equal angles with **OX, OY**. [Produce **XO** to **X′**; bisect ∠ **X′OY**; through **A** draw a line ‖ to bisector.]

CHAPTER VI.

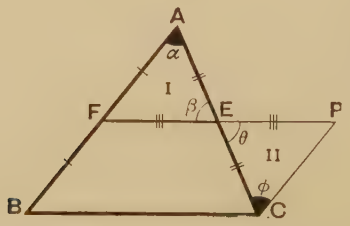
ADDITIONAL PROPOSITIONS ON TRIANGLES.

PROP. 28.

The straight line which joins the middle points of two sides of a triangle is parallel to the third side, and is equal to one-half of it.

Let ABC be the △.

Given: E bisects AC, F „ AB } **Required to prove**: EF || to BC.



Proof.—Produce FE making EP = EF.

AE = EC (hypothesis)
FE = EP (construction)
β = θ
} ∴ △ AEF ≡ △ PEC

∴ PC = AF, which = FB

Also α opposite FE in I = φ opposite EP in II

∴ PC, AB are || (AC makes equal alternate ∠ s with them)

i.e. PC, FB are || and equal
∴ FP, BC „ „
i.e. FE is || to BC
Also FP was made = 2FE
∴ BC = 2FE

Conversely.—*If a line be drawn through the middle point of one side parallel to a second side, it bisects the third side.*

For only one straight line can be drawn through F || to BC
and „ „ „ „ „ F to bisect AC

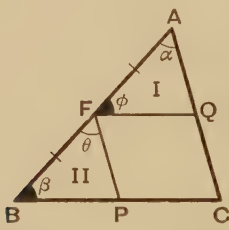
We have proved in the proposition that the second of these lines is identical with the first; hence no further proof is needed to show that the first is identical with the second.

Prop. 29.

(Converse of Prop. 28.) Independent Proof.

Given: F bisects AB
FP || to AC
FQ || to BC

Required to prove: { P bisects BC
Q „ CA



Proof.—AF = BF (hypothesis)
$\alpha = \theta$ (AB cuts ‖s AC, FP)
$\phi = \beta$ („ FQ, BC)
$\therefore$ I ≡ II

$\therefore$ FP = AQ

Also FP = QC (FPCQ is ▱)
$\therefore$ AQ = QC *i.e.* FQ bisects AC
similarly FP „ BC

Exercises.

1. Two triangles are on the same base and between the same parallels: prove that a parallel to the base through the middle point of one side bisects the other three sides.

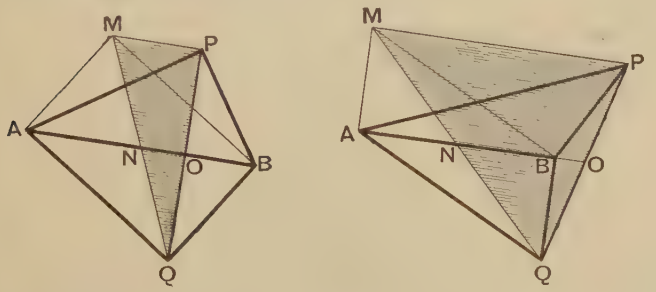
2. Show that when the middle points of the sides of any quadrilateral are joined a ▱ is formed, and prove that the area of this parallelogram is half the area of the quadrilateral.

3. If the middle points of the diagonals of any quadrilateral be joined to the middle points of two opposite sides, the figure so formed is a ▱.

PROP. 30.

If two triangles of equal area have the same base and lie on opposite sides of it, the line joining their vertices is bisected by the common base.

Let **PAB, QAB** be △s of equal area.
Join **PQ**, cutting **AB** at **O.**



Required to prove: **PO = OQ.**

Proof.—Draw **BM** || to **QA**, **AM** || to **QB** } let them cut at **M.**

AQBM is ▱ ∴ **AB** bisects it

∴ △ **MAB** = △ **QAB** = △ **PAB**

∴ **MP** is || to **AB** (Prop. 26, Cor. 4)

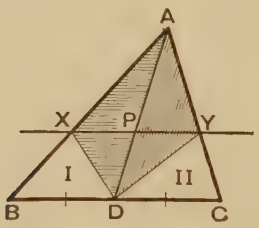
Also **N** is middle point of **MQ** (diagonals of ▱ bisect each other);
and **NO** drawn through it is || to side **MP** of △ **QMP**;
∴ **O** bisects **PQ** (Prop. 29)

Definition.—The lines joining the **middle points of the sides** of a △ to the **opposite corners** are called **medians.**

Prop. 31.

If a line be drawn parallel to a side of a triangle, the part of it cut off by the other sides is bisected by the corresponding median.

Let **XY** be || to side **BC** of △ **ABC**.
Let median **AD** cut it at **P**.



Required to prove: **PX = PY.**

Proof.—△ **BDA** = △ **CDA** (equal bases and same height)

I = II ,, ,, ,,

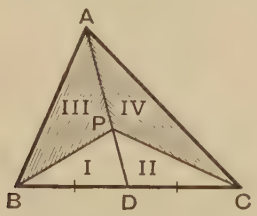
∴ △ **ADX** = △ **ADY** (by subtraction)

they are on opposite sides of the same base **AD**;

∴ **XY** is bisected by **AD** (Prop. 30)

PROP. 32.

If two vertices B, C *of a triangle* ABC *be joined to any point* P *on the median through the third vertex* A, *the triangles* III, IV *thus formed are equal in area.*



Proof.—$\triangle$ ABD = $\triangle$ ACD (same height, and base BD = base CD)
I = II " " " "
$\therefore$ III = IV (by subtraction)

Converse.—*If equal triangles* III *and* IV *be formed on the bases* AB, AC *by joining a point* P *to the corners of a triangle* ABC, *then* P *lies on the median through* A.

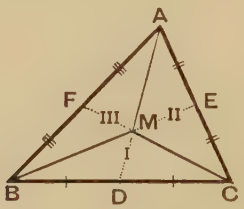
This is merely a different enunciation of Prop. 30, where B is proved to be a point on the median AO of $\triangle$ APQ.

Definition.—Lines which meet at a point are said to be **concurrent.**

PROP. 33.

The medians of a triangle are concurrent.

Let M be the point of intersection of two medians BE, CF of △ ABC.



Required to prove : M lies on the median AD.

Proof.—M is on median BE ∴ III = I (Prop. 32)

M „ „ CF ∴ II = I

∴ III = II

∴ M lies on the median AD

i.e. the three medians are concurrent at M.

EXERCISE.

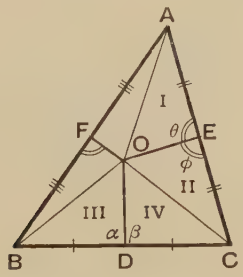
In the figure of Prop. 33 prove that AD = 3DM.

PROP. 34.

The straight lines drawn at right angles to the sides of a triangle through their middle points are concurrent.

Let D, E, F be the middle points of the sides of △ ABC.

The ⊥s to AB at F and AC at E, being ⊥ to intersecting lines AB, AC, must meet at some point O.



Required to prove: OD is ⊥ to BC.

Proof.—AE = EC (hypothesis)
OE = OE
θ = φ (right ∠s)
} ∴ I ≡ II

∴ OC = OA

Similarly OB = OA

∴ OB = OC

OB = OC (proved)
OD = OD
BD = DC (hypothesis)
} ∴ III ≡ IV

∴ α = β

∴ each is a right ∠; *i.e.* OD is ⊥ to BC

i.e. the three ⊥s to sides at D, E, F are concurrent.

Exercises.

1. The quadrilateral **PQRS** is bisected by its diagonal **QS**: prove that **QS** bisects **PR**.

2. Prove that the bisectors of the angles of a $\triangle$ are concurrent.

3. Prove that the perpendiculars drawn to the sides of a $\triangle$ from the opposite corners are concurrent.

[Note.—Draw through each corner a line $\parallel$ to opposite side, and use Prop. 34.]

4. **ABCD** is a parallelogram; **AB** is bisected in **E**, and **CD** in **F**; **DE**, **BF** are drawn: prove that these lines divide **AC** into three equal parts.

5. **ABCD** is a parallelogram; **X** is the middle point of **AD**; **BX** cuts **AC** at **Y**: show that $\mathbf{CY} = 2\mathbf{AY}$.

6. **ABCD** is a quadrilateral; **P** is middle point of **AB**, **Q** is middle point of **CD**: if **PQ** bisect **BD**, prove that **AD** is parallel to **BC**.

CHAPTER VII.

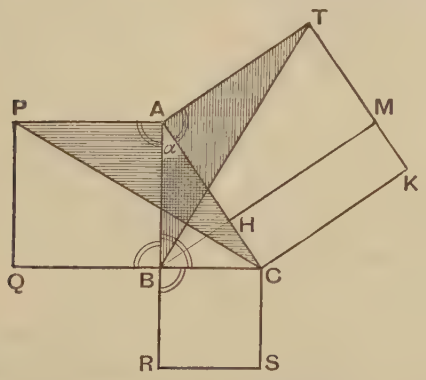
RIGHT-ANGLED TRIANGLES.

PROP. 35. [**Euclid I. 47.**]

The square constructed on the hypotenuse of a right-angled triangle is equal to the sum of the squares constructed on the other two sides.

Let △ ABC have a rt. ∠ at B.

On AB, BC, CA construct squares AQ, BS, CT.



Required to prove: AQ + BS = CT.

Proof.—ABQ, ABC are rt. ∠s; ∴ CBQ is a st. line (Prop. 2).

CBR, CBA „ „ ∴ ABR „ „

Let BHM, || to AT, cut AC at H and TK at M. Join PC, BT.

△ PAC and sq. AQ have same "base," PA, same "height," AB;

∴ square AQ = twice △ PAC

△ TAB and rect. AM have same "base," AT, same "height," AH;

∴ rectangle AM = twice △ TAB

Also—

$$\triangle PAC \equiv \triangle TAB, \because \begin{cases} PA = AB \text{ (sides of square)} \\ CA = AT \quad „ \quad „ \\ \angle PAC = \angle TAB \text{ (each = rt. } \angle + \alpha) \end{cases}$$

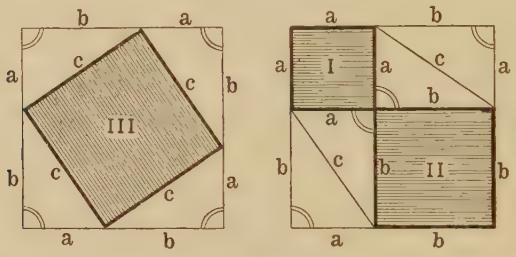
∴ sq. AQ (which is 2 △ PAC) = rect. AM (which is 2 △ TAB)

Similarly we prove that square BS = rectangle CM;

∴ sq. AQ + sq. BS = rect. AM + rect. CM = sq. CT

Many alternative proofs of this important proposition have been devised. We give one of the most interesting.

Each of the figures is a square constructed on a line $(a + b)$; in each the same right-angled $\triangle$ with sides **a**, **b** and hypotenuse **c** is repeated four times.



In the left figure, equal lengths **a** are cut from each side in turn, leaving equal remainders **b**. The points in the sides are joined: the four $\triangle$s formed are equal (by Prop. 9), and each has hypotenuse **c** of same length. The $\angle$s of **III** are easily proved to be rt. $\angle$s.

$\therefore$ **III** is square on **c**.

In the right figure, equal lengths **a** are cut from the sides at one corner, and lengths **b** from the sides at opposite corner. The squares **I**, **II** being constructed leave as complements the rectangles whose sides are **a**, **b**.

Take the four $\triangle$s from the big square on left; remainder is **III**.

Take the four equal $\triangle$s from the big square on right; remainder is **I** + **II**.

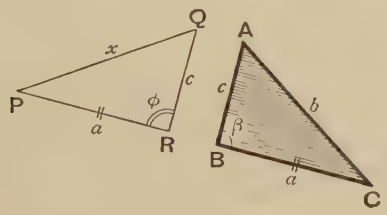
$$\therefore \text{III} = \text{I} + \text{II}$$

NOTE.—In future we shall write AB^2 for "the square constructed on **AB**," and b^2 for "the square constructed on **b**."

PROP. 36. [**Euclid I. 48.**]

If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by the latter is a right angle.

Let **ABC** be the △.



Given : $b^2 = a^2 + c^2$.

Required to prove : β is a rt. ∠.

Proof.—Let **PRQ** be a △, having a rt. ∠ ϕ, and also having **PR** = **CB** = a, and **QR** = **AB** = c;

$\therefore x^2 = a^2 + c^2$ (ϕ is a rt. ∠)

also $b^2 = a^2 + c^2$ (hypothesis)

$\therefore x^2 = b^2$

$\therefore x = b$

Now—

QR = **AB**
PR = **CB**
PQ = **CA**
$\therefore$ △ **PRQ** ≡ △ **CBA**

$\therefore \phi = \beta$

But ϕ is a rt. ∠;

$\therefore \beta$ is a rt. ∠

Since the area of a square is measured by "squaring" the length of a side, Prop. 35 enables us to calculate the length of the third side when any two sides of a right-angled $\triangle$ are known. In the following examples we shall denote the hypotenuse by **c**, and the other sides by **a**, **b**.

$$c^2 = a^2 + b^2$$

EXAMPLE 1.—Given $a = 3$ ft., $b = 4$ ft.;

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25 ; \therefore c = 5 \text{ ft.}$$

EXAMPLE 2.—Given $c = 65$ ft., $a = 63$ ft.: $65^2 = 63^2 + b^2$;

$$\therefore b^2 = 65^2 - 63^2 = (65 + 63)(65 - 63) = 128 \times 2 = 256$$

$$\therefore b = 16 \text{ ft.}$$

EXAMPLE 3.—Given $c = 89$ in., $b = 43$ in.: $89^2 = a^2 + 43^2$;

$\therefore a^2 = 89^2 - 43^2 = (89 + 43)(89 - 43) = 132 \times 46 = 6072$ sq. in.

$\therefore a = \sqrt{6072}$
$= 77{\cdot}92$ in.

132	
46	
528	
792	
6072	

	60′72′(77·92
147	1172
154	1430
	44

In this example, 6072, the value of a^2, is not a "perfect" square, *i.e.* there is no number which, when multiplied by itself, gives the result 6072 exactly; hence the length of **a** cannot be expressed in inches *exactly*. But by working the square root by the method of approximation, we find a result as close as we please to the value of **a**. Thus $a = 77{\cdot}92$ in., within 0·01 of an inch.

EXERCISES.

In the following exercises on right-angled triangles, **a**, **b** denote the sides containing the right angle, **c** denotes the hypotenuse.

1. Given $a = 12$ cms., $b = 5$ cms., calculate **c**.
2. „ $c = 1$ ft. 5 in., $b = 8$ in., calculate **a**.
3. „ $a = 12$ ft., $b = 19$ ft., calculate **c**.
4. „ $a = 19{\cdot}3$ cms., $b = 44{\cdot}8$ cms., calculate **c**.
5. „ $b = 416$ miles, $c = 632$ miles, calculate **a**.

6. Given **a** = 561 metres, **c** = 817 metres, calculate **b**.
In each of the above the result is to be calculated to three significant figures.

7. A ladder reaches across a moat 19 ft. wide to a window 32 ft. above the edge of the moat: calculate length of ladder to nearest inch.

8. Two boys 56 yards apart are flying a kite; it is directly over the head of one of them; 119 yards of string are out: if the string were quite straight (in practice it must sag a little), what would be the height of the kite?

9. Two straight roads cross at right angles; a man runs from the corner 7 miles along one road, while a motor travels 24 miles along the other: how far are they then apart?

10. If **O** is the point of intersection of the perpendiculars from **A, B, C** on opposite sides of a $\triangle$ **ABC**, prove that—

$$OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2.$$

11. If the sides of a $\triangle$ are 2 cms., 3 cms., 4 cms., is it acute-angled, obtuse-angled, or a right-angled $\triangle$?

12. The diagonals of a quadrilateral intersect at rt. $\angle$s: show that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

13. Describe a square having its area double the area of a given square.

14. If squares be described on the sides of any $\triangle$ **ABC**, prove that the perpendicular from **A** on **BC** divides the square on **BC** into two parts which differ from the squares on **AB** and **AC** by equal areas.

15. Describe a square which is three times a given square.

16. Construct a square equal to the sum of two given squares. Also construct a square equal to the difference of two given squares.

17. In figure of Prop. 35 prove that **AQ** is $\parallel$ to **RC**.

18. **ABCD** is a convex quadrilateral having a rt. $\angle$ at **B**. If the sum of the squares on **AB, BC, CD** be equal to the square on **AD**, prove that **ACD** is a rt. $\angle$.

19. **BCDE** is the square on hypotenuse of the right-angled $\triangle$ **ABC**, and **CAFG** is the square on side **CA**, both squares being external to the $\triangle$; **X** is any point on **CA**, and **XG** and **XD** are joined: prove that $\triangle$ **GCX** = $\triangle$ **DCX**.

20. Squares are described upon the three sides of any triangle, the squares being all outside the $\triangle$, and adjacent corners of different squares are joined: show that the four $\triangle$s in the figure so formed are all equal in area.

21. A ladder placed flat against a vertical wall just reaches to the top; when the foot of the ladder has been pulled out 24 ft. the top has slipped 18 ft. down the wall: find the length of the ladder.

22. The △ ABC has a rt. ∠ at A; ABFG, ACHK are squares on AB and AC, both external: prove that the centres of these squares and the middle points of BC, GK are corners of a square.

Miscellaneous Exercises.

1. Describe a △, having given the vertical angle, the difference of the angles at the base, and the less of the two sides.

2. A △ has each of its sides double the corresponding side of another △: determine the relation between the angles and between the areas of the two △s.

3. If a △ be constructed with its sides equal to three given straight lines, show that if the square on one of these lines exceeds the sum of the squares on the other two, the △ has an obtuse ∠.

4. If P be a point on the diagonal AC of a ▱ ABCD, prove that the △ PBD is equal to half the difference of the ▱s about AC formed by parallels drawn through P.

5. ABC is any △; on the sides AC, BC squares ACDE, BCFH are described: show that the lines AF, BD are equal.

6. Describe a right-angled △ equal in area to a given △ and having one of its sides perpendicular to a given straight line.

7. If ABCD be a ▱, and from K, a point on the diagonal AC, EKF be drawn parallel to AD to meet AB in E and DC in F, and HKG be drawn parallel to AB to meet AD in H and BC in G, prove that △ AGF + △ AEH = △ ABC.

8. The sides of a triangle ABC are produced as follows:—BC to D, CA to E, and AB to F, so that BD = 3BC, CE = 3CA, and AF = 3AB: prove that △ DEF = 19 △ ABC.

9. C is the right angle of a right-angled triangle ABC; D is taken in BC so that CAD is an isosceles triangle: show that if BDA is also isosceles, the angle A is three times the angle B.

10. From every point of a given line, the lines drawn to each of two given points on opposite sides are equal: prove that the line joining the given points is at right angles to the given line.

11. The sides of a triangle are respectively a = 9 cms., b = 7 cms., c = 8 cms.; a perpendicular is let fall upon c from the opposite vertex: find the lengths of the segments of c.

12. PQRS is a quadrilateral figure having the opposite sides

PQ and **RS** equal to one another, and the opposite angles **PQR** and **PSR** supplementary : prove that **PS** is parallel to **QR**.

13. **ABC** is a $\triangle$: find a point **P** in the side **AC** so that if **PX** and **PY** are drawn parallel to **AB** and **BC** respectively, meeting the sides in **X** and **Y**, then **PX** = **PY**.

14. Through a given point **P** between two given straight lines draw a straight line terminated by the given lines and bisected at **P**.

15. **ABCD** is a quadrilateral having **AB** = **CD** ; when **BD** is joined $\angle$ **ABD** + $\angle$ **BDC** = 2 rt. $\angle$s : prove $\triangle$ **ABD** = $\triangle$ **CBD**.

16. The point **P** is given in position within the angle **AOB** between two infinite straight lines **OA** and **OB**, and the line **QPR** is drawn through **P**, cutting **OA** and **OB** in **Q** and **R** : prove that if the area of the triangle **QOR** is the least possible, it is bisected by **OP**.

17. The opposite sides **AD**, **BC** of a quadrilateral **ABCD** are parallel ; **E** is the intersection of the diagonals, and **F** is the middle point of **BC** : prove that the triangles **AEF**, **DEF** are equal.

18. The straight line which joins the middle points of two opposite sides of a quadrilateral bisects the figure : prove that the bisected sides are parallel.

ANSWERS TO EXERCISES

Section I. Part I.

Page 21.

1. 120°, 30°. **2.** 45°. **3.** 112° 30′. **4.** 84°. **5.** 9°

Page 23.

1. 149°, 112°, 107° 37′, 60°, 28°. **2.** 163° 30′, 122°, 62° 15′, 5°.
3. 75°, 9°, 66° 33′.

Page 29.

1. 45°. **2.** (1) 72° ; (2) 60°. **3.** (1) 120° ; (2) 144°,
4. 160°. **5.** Nine. **6.** 20°.

Page 32.

1. 35°. **2.** 107°. **3.** 35°. **4.** 40° 55′.
5. $\alpha = \beta = 72°$, $\gamma = 36°$. **6.** $\alpha = 22° 30'$, $\beta = 67° 30'$, $\gamma = 90°$.
7. 18°. **8.** 58° or 22°.

Page 42.

9. Eight.

Page 63.

37. Sixteen.

Page 78.

1. 529 sq. cms.
2. 108 sq. cms.
3. 1430 sq. cms.
4. 3·346 acres.
5. 102·9 sq. in.
6. 12·34 sq. metres.
7. 22·1 cms.
8. 63·46″.

Page 80.

8. Seven.

Page 84.

1. (1) A = 29°, B = 46·6°, C = 104·5°.
   (2) A = 44·4°, B = 63°, C = 72·6°.
   (3) A = 37·8°, B = 88·5°, C = 53·7°.
   (4) A = 82·6°, B = 62·9°, C = 34·5°.
   (5) and (6) impossible.
4. Diag. = 3·32″, ∠s 79·4°, 100·6°.
5. Diag. = 6·22 cms., ∠s 98·8°, 81·2°.
6. Diag. = 3·50 cms., A = 89·3°, B = 137·7°, C = 60°, D = 73°.

Page 88.

6. 3·53″; 3·09 sq. in.

Page 90.

1. 0·726″, 0·726 sq. in.
2. 1·96″, 2·94 sq. in.
3. 3·8 cms., 9·5 sq. cms.
4. 4·36 cms., 6·10 sq. cms.

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2. 17·72 sq. cms.
3. C = 77·2°, D = 122·7°; 2·14 sq. in.
15. 70·2°.

Page 112.

1. 13 cms.
2. 1′ 3″.
3. 22·5′.
4. 48·8 cms.
5. 476 miles.
6. 594 metres.
7. 37′ 3″.
8. 105 yards.
9. 25 miles.
21. 25′.

Page 114.

11. 6 cms., 2 cms.

Alternative Treatment of Parallels.

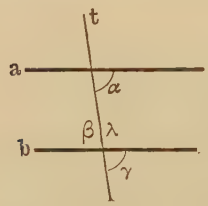
Note.—The following alternative definition and pair of propositions are inserted *purely for examination purposes* for the benefit of candidates for the Previous Examination in the University of Cambridge and for some other examinations. While the Cambridge regulations admit any proof which forms "part of a systematic treatment of the subject," the enunciation of Propositions concerning Parallels given in the schedule unfortunately *implies* the definition of parallels by their negative non-meeting property; this course seems to the author less logical than that adopted in the text (pp. 35–49).

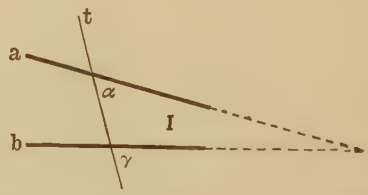
Definition.—Two **straight lines** in the **same plane** are said to be **parallel** if they **do not meet** when produced in either direction to any **finite distance** however great.

Prop. 6. [Euclid I. 27, 28.]

If one of the interior angles made by two straight lines with any transverse line be equal to the corresponding exterior angle, the former lines are parallel.

Let **a**, **b** be two straight lines, making equal ∠s α, γ with **t**.
Required to prove: **a**, **b** cannot meet at a finite distance.





Proof.—One of the following statements must be true, and only one can be true:—

either (1) **a**, **b** do meet at a finite distance,
or (2) „ do not.

If **a**, **b** were to meet when produced, say, to the right, then **I** would be a △; and γ (exterior ∠) would be greater than α (interior non-adjacent ∠); but it is not.

∴ **a**, **b** do not meet towards the right
similarly, **a**, **b** „ „ „ left
∴ (1) is not true,
∴ (2) must be true,

i.e. **a**, **b** do not meet when produced to a finite distance;
i.e. **a**, **b** are parallel.

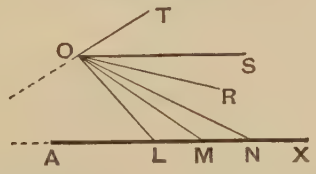
Corollary 1.—If an **interior** ∠ α be **equal** to the **alternate interior** ∠ β, the lines **a**, **b** are ||.
(For $\beta = \gamma$ ∴ $\alpha = \gamma$ ∴ **a**, **b** are ||)

Corollary 2.—If the sum of **two interior** ∠s α, λ on the same side of **t** be **two right angles**, the lines **a**, **b** are ||.
(For $\gamma + \lambda = 2$ rt. ∠s, and by supposition $\alpha + \lambda = 2$ rt. ∠s,
∴ $\gamma + \lambda = \alpha + \lambda$ ∴ $\gamma = \alpha$ ∴ **a**, **b** are ||.)

Playfair's Axiom.

Consider a straight line **AX**, and draw a set of lines **OL**, **OM**, etc., through the same point **O**.

In drawing **OL** the straight-edge is inclined fairly steeply to **AX**. Then to draw **OM**, **ON**, etc., it must be rotated so that its inclination decreases, and so that it points more nearly



in the same direction as **AX**. At the same time the points of intersection, **L**, **M**, **N**, recede from **A**.

OR does not meet **AX**, but we see that it would if both were produced far enough.

OS, which points in the same direction as **AX**, and is parallel to it, does not meet it however far produced.

OT would meet **AX** if both were produced backward.

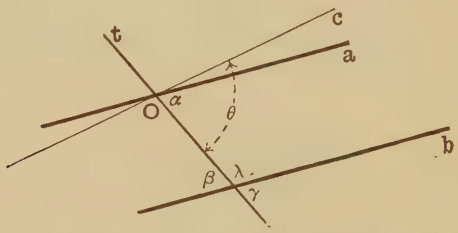
We shall assume, as evident from experience, that every line through **O**, except **OS**, meets **AX**, if both be produced far enough one way or the other; that is, we assume as an **axiom—**

There cannot be drawn through the same point more than one line parallel to the same line.

Note.—This axiom is known as **Playfair's axiom.**

PROP. 7 (CONVERSE OF PROP. 6). **[Euclid I. 29.]**

If a transverse straight line cross two parallel straight lines, it makes the alternate angles equal.



Let a, b be || straight lines, cut by a transverse line t.

Required to prove: $\alpha = \beta$.

Proof.—One of the following statements must be true, and only one can be true :—

either (1) $\alpha = \beta$,
or (2) α is not equal to β.

There must be some st. line through O making with t an $\angle$ equal to β; if possible, let c, which is distinct from a, make $\theta = \beta$.

$\therefore$ c is || to b,

$\therefore$ both c and a are drawn through O || to b; this is impossible (Playfair's axiom).

$\therefore$ (2) is not true,

$\therefore$ (1) is true, *i.e.* $\alpha = \beta$.

Corollary 1.—Exterior $\angle$ γ = corresponding interior $\angle$ α.
(For $\gamma = \beta$, and $\beta = \alpha$.)

Corollary 2.—The sum of the **interior** $\angle$s on the same side of the transverse line is **2 rt. $\angle$s**, *e.g.* $\alpha + \lambda = 2$ rt. $\angle$s.
(For $\gamma + \lambda = 2$ rt. $\angle$s, and $\gamma = \alpha$.)

END OF SECTION I. PART I

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GEOMETRY

CHAPTER VIII.

CIRCLES.

Definitions.

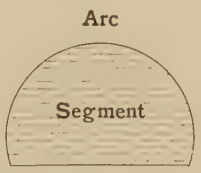
Any part of the circumference of a circle is called an **Arc.**

The straight line joining any two points on the circumference is called a **Chord.**

Any part of a circle cut off by a chord is called a **Segment.**

Any part of a circle bounded by an arc and two radii is called a **Sector.**

Circles which have the same centre are said to be **concentric.**



We will first state theorems concerning circles which are so obviously true that we do not include them in the propositions formally proved.

(1) A circle cannot have more than one centre [for if a circle had two centres, the chord which passed through them both would have two middle points].

(2) Two concentric circles cannot meet or cut one another [for if their radii be equal, they coincide altogether, *i.e.* they are one circle, not two; and if their radii be unequal, every point on the circle of greater radius lies outside the other circle].

(3) If two circles have equal radii, they are equal in all respects [for if one be placed on the other, centre to centre, they coincide].

(4) If two circles be equal in area, their radii are equal [for if their radii were unequal, one would lie entirely within the other, when placed upon it, centre to centre].

Prop. 37. [Euclid III. 3.]

(1) *If a diameter of a circle bisect a chord which is not a diameter, the diameter is at right angles to the chord.*

(2) **Converse.**—*If a diameter be at right angles to a chord which is not a diameter, the diameter bisects the chord.*

(1) **Given**: Diameter CD bisects chord AB at N.
Required to prove: CD $\perp$ to AB.

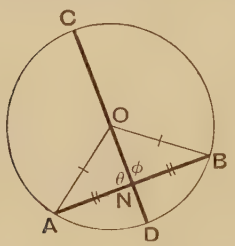


Fig. 1.

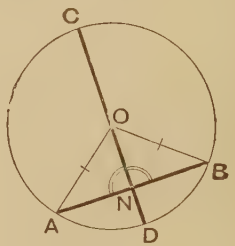


Fig. 2.

Proof.—Join A and B to centre O.

AO = BO (radii)
AN = BN (hypothesis)
ON = ON
$\therefore \triangle$ AON $\equiv \triangle$ BON

$\therefore \theta$ opposite OA = ϕ opposite OB

θ, ϕ are adjacent $\angle$s; $\therefore$ each is a rt. $\angle$

$\therefore$ CD is $\perp$ to AB

(2) **Given**: Diameter CND $\perp$ to chord AB.
Required to prove: AN = BN.

Proof.—Join A and B to centre O.
In right-angled $\triangle$s OAN, OBN—
hypotenuse OA = hypotenuse OB
side ON = side ON
$\therefore$ $\triangle$s are equal in all respects (Prop. 14)
$\therefore$ AN = BN

Symmetry.

If the circle in Prop. 37 be folded about the diameter **DOC**, **NA** will lie along **NB** (since rt. $\angle \theta$ = rt. $\angle \phi$); and since **NA** = **NB**, the point **A** will lie on the point **B**.

This is true for any chord $\perp$ to **DOC**.

Hence, when folded, the one semicircle coincides with the other, and therefore a semicircle is truly half the circle.

On account of the symmetrical way in which points such as **A**, **B** lie at equal distances on opposite sides of **DOC**, a **diameter** is called a **line of symmetry of the circle.**

Note.—**If** we accept as an **axiom** that "every diameter of a circle is a line of symmetry," it is sufficient proof of Prop. 37 to say that "the two parts **AN**, **BN** of any chord, perpendicular to the diameter **DOC**, coincide when the circle is folded, and are therefore equal."

The following are examples of lines of symmetry:—

(1) In an isosceles $\triangle$, the bisector of vertical $\angle$.

(2) In a square or a rhombus, each diagonal.

(3) In a rectangle, each line bisecting two opposite sides.

Exercises.

1. If one diagonal of a quadrilateral be a line of symmetry, what sides must be equal?

2. If both diagonals of a quadrilateral be lines of symmetry, what kind of quadrilateral must it be?

3. **AOC** and **BOD** are diameters of a $\odot$ **ABCD**: prove that if **AB** and **CD** are joined, these two lines are parallel.

4. Describe a $\odot$ with a given centre cutting a given $\odot$ at the extremities of a diameter.

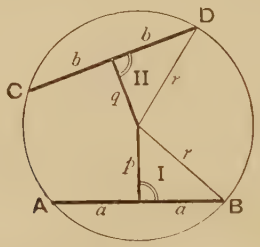
5. From the ends of a diameter **BC** of a circle parallel chords **BE**, **CF** are drawn meeting the circle again in **E** and **F**: prove that **EF** passes through the centre of the circle.

Prop. 38. [Euclid III. 14.]

(1) *Equal chords of a circle are equally distant from the centre.*

(2) **Converse.**—*Chords which are equidistant from the centre are equal.*

(1) **Given**: Chord AB = chord CD.
Required to prove:
$\perp p$ from centre to AB = $\perp q$ from centre to CD.



Proof. p, q bisect AB, CD, which are equal.

$$\therefore a = b$$

In right-angled $\triangle$s I, II—

$$\left.\begin{array}{l}\text{hypotenuse } r = \text{hypotenuse } r \\ \text{side } a = \text{side } b\end{array}\right\} \therefore \text{I} \equiv \text{II}$$

$$\therefore p = q$$

(2) **Given**: $\perp p = \perp q$.
Required to prove: AB = CD.

Proof.—In right-angled $\triangle$s I, II—

$$\left.\begin{array}{l}\text{hypotenuse } r = \text{hypotenuse } r \\ \text{side } p = \text{side } q\end{array}\right\} \therefore \text{I} \equiv \text{II}$$

$$\therefore a = b \quad \therefore 2a = 2b$$

i.e. AB = CD

Prop. 39. [Euclid III. 15.]

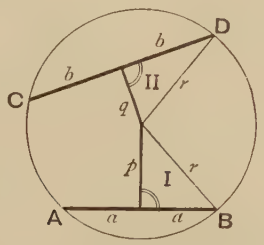
(1) *If one of two chords of a circle be nearer to the centre, it is the greater chord.*

(2) **Converse.**—*If one of two chords be greater than the other, it is nearer to the centre.*

(3) *A diameter is greater than any chord which is not a diameter.*

(1) **Given:** $\perp p$ from centre to chord AB $>$ $\perp q$ from centre to chord CD.

Required to prove: AB $<$ CD.



Proof. $a^2 + p^2 = r^2$ ($\because$ I is right-angled $\triangle$)

$b^2 + q^2 = r^2$ ($\because$ II „ „)

$\therefore a^2 + p^2 = b^2 + q^2$

But $p^2 > q^2$ (since $p > q$) and $\therefore a^2 < b^2$

$\therefore a < b$

Also p, q bisect AB, CD; *i.e.* AB $= 2a$, CD $= 2b$;

$\therefore$ AB $<$ CD

(2) **Given:** AB $<$ CD.

Required to prove:

$\perp p$ from centre on AB $>$ $\perp q$ from centre on CD.

Proof. p, q bisect AB, CD; $\therefore a < b$.

$a^2 + p^2 = b^2 + q^2$ (proof as before)

But $a^2 < b^2$ (since $a < b$) and $\therefore p^2 > q^2$

$\therefore p > q$

(3) The two radii drawn to the ends of a chord are together greater than the chord (for the chord and radii make a $\triangle$); $\therefore$ any diameter is greater than the chord.

Prop. 40. [Euclid III. 20.]

In a circle, the angle at the centre which stands on any arc is double any angle at the circumference standing on the same arc.

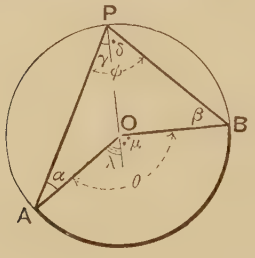


Fig. 1.

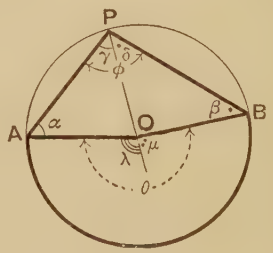


Fig. 2.

Let θ be the $\angle$ at the centre **O** } standing on same
ϕ „ „ circumference **P** } arc **AB**

Required to prove: $\theta = 2\phi$.

Proof.—Join **PO**; it may lie (1) between **PA, PB**;
(2) not between **PA, PB**.

(1) **PO** between **PA, PB**, as in Figs. 1 and 2.

Produce **PO**, making $\angle \lambda$ external to $\triangle$ **AOP**; $\therefore \lambda = \alpha + \gamma$

$\angle \mu$ „ „ $\triangle$ **BOP**; $\therefore \mu = \beta + \delta$

$\triangle$ **AOP** is isosceles; $\therefore \alpha = \gamma \quad \therefore \lambda = 2\gamma$

$\triangle$ **BOP** „ $\therefore \beta = \delta \quad \therefore \mu = 2\delta$

$\therefore \theta = \lambda + \mu = 2\gamma + 2\delta = 2(\gamma + \delta)$

i.e. $\theta = 2\phi$

(2) PO not between PA, PB (Fig. 3).

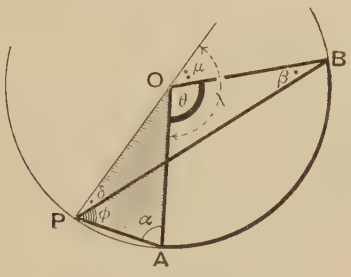


FIG. 3.

Produce PO, making $\angle \lambda$ external to $\triangle$ AOP,
and $\angle \mu$ „ $\triangle$ BOP;
$\therefore \lambda = \alpha + (\delta + \phi)$
and $\mu = \beta + \delta$

$\triangle$ AOP is isosceles; $\therefore \alpha = (\delta + \phi)$ $\therefore \lambda = 2(\delta + \phi)$
$\triangle$ BOP „ $\therefore \beta = \delta$ $\therefore \mu = 2\delta$

Subtract; $\therefore \lambda - \mu = 2\phi$
i.e. $\theta = 2\phi$

EXERCISE.

Two equal circles intersect in A and B, and any straight line through A meets the circles again in P and Q: prove that P and Q are equidistant from B.

Definition.—Angles, such as α, β, γ, formed by lines drawn to any point of the arc from the ends of the chord which cuts off a segment, are called **angles in the segment**; they are also said to **stand on the arc** which has been cut off below the segment, or to be **subtended by** the same arc.

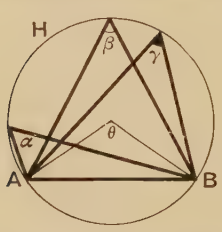
EXERCISES.

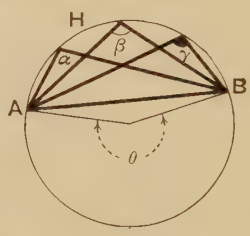
1. Determine the shortest chord that can be drawn through a given point inside a ⊙.

2. Draw a circle 2″ radius; cut off a segment; draw several $\angle$s α, β, γ in it; measure them with a protractor.

Prop. 41. [Euclid III. 21.]

All angles in the same segment of a circle are equal.





Let α, β, γ be angles in the segment ABH cut off by chord AB.

Required to prove : $\alpha = \beta = \gamma$.

Proof.—Join A and B to centre, making $\angle\ \theta$.

$$\left.\begin{aligned}\theta &= 2\alpha\\ \theta &= 2\beta\\ \theta &= 2\gamma\end{aligned}\right\} \therefore\ \alpha = \beta = \gamma$$

Alternative Proof of Prop. 45 (for enunciation see p. 129).

In left-hand figure above, where segment AHB is greater than a semicircle,

$$\angle\ \theta \text{ at centre} < 2 \text{ rt. } \angle\text{s};$$

$$\text{also each of } \alpha, \beta, \gamma = \tfrac{1}{2}\,\theta$$

$$\therefore \quad \text{,,} \quad \alpha, \beta, \gamma < 1 \text{ rt. } \angle$$

In right-hand figure, where segment AHB is less than a semicircle,

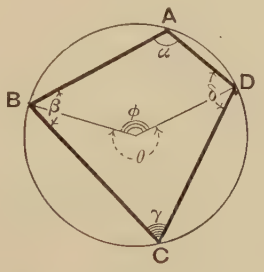
$$\angle\ \theta \text{ at centre} > 2 \text{ rt. } \angle\text{s}$$

$$\therefore \text{ each of } \alpha, \beta, \gamma > 1 \text{ rt. } \angle$$

Draw a figure having segment AHB a semicircle; in it θ will be 2 rt. $\angle$s, hence in it α, β, γ will each be 1 rt. $\angle$.

Prop. 42. [Euclid III. 22.]

The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.



Let α, β, γ, δ be the $\angle$s of quadrilateral **ABCD** inscribed in a circle.

Required to prove: $\alpha + \gamma = 2$ rt. $\angle$s.

Proof.—Draw radii to **B** and **D**, making $\angle$s θ and ϕ at centre.

$$\left.\begin{aligned}\theta &= 2\alpha \\ \phi &= 2\gamma\end{aligned}\right\} \therefore \theta + \phi = 2(\alpha + \gamma)$$

But $\theta + \phi = 4$ rt. $\angle$s

$\therefore \alpha + \gamma = 2$ rt. $\angle$s

Similarly $\beta + \delta = 2$ rt. $\angle$s

Exercises.

1. Through any point **O** on the side **BC** of an equilateral $\triangle$ **ABC** the straight lines **OK**, **OL** are drawn parallel to **AB**, **AC** respectively, to meet **AC**, **AB** respectively in the points **K** and **L**; and the circle through **O**, **K**, and **L** cuts **AB**, **AC** again in the points **P** and **Q**: show that **OPQ** is an equilateral $\triangle$.

2. **AB** and **CD** are equal chords of a circle: prove that either **AC** is $\parallel$ to **BD**, or **AD** $\parallel$ to **BC**.

3. **AB** is any chord of a circle whose centre is **O**, it is produced to **C** so that **BC** = **AO**; **CO** is joined cutting the circle at **D**, and is also produced to cut it again at **E**: show that $\angle$ **AOE** = three times $\angle$ **BOD**.

4. Prove that any $\square$ inscribed in a circle is a rectangle.

5. **ABCD** is a quadrilateral inscribed in a circle, and **AB** = **CD**: show that **AC** = **BD**, and that **AD** and **BC** are $\parallel$.

6. Two arcs of circles are described on the same side of a straight line **AB**; a line **APQ** is drawn through **A** to meet the arcs in **P** and **Q**: prove that $\angle$ **PBQ** is of the same magnitude for all positions of the chord.

7. If **ABCD** be a quadrilateral inscribed in a circle, and the bisector of the angle **BAD** meet the circumscribed circle in **E**, prove that **EC** bisects the angle between **DC** and **BC** produced.

8. **ABC** is an acute-angled $\triangle$; perpendiculars **AL** and **BM** are drawn from **A** and **B** to the opposite sides, meeting in **O**: prove that $\angle$ **COL** = $\angle$ **CML**.

9. Let **A** and **B** be two points on a circle **ADB**, whose centre is **O**; let an arc of a circle be described through **AOB**; let any straight line **APQ** be drawn cutting the arc **AOB** at **P** and **ADB** at **Q**: then prove **PB** = **PQ**.

Exercises on Prop. 38.

1. A chord, length 5′ 10″, is drawn in a $\odot$, radius 3′ 1″: what is its distance from centre?

$a = \frac{1}{2}$ chord $= 35''$, $r = 37''$ $\therefore p^2 = 37^2 - 35^2 = 2 \times 72 = 12^2$ $\therefore p = 12''$

2. A chord of a $\odot$, radius 52 cms., is at a distance 11·2 cms. from the centre: find length of chord to nearest millimetre.

$$a^2 = 52^2 - 11{\cdot}2^2 = 63{\cdot}2 \times 40{\cdot}8 = 2578{\cdot}4$$

$$\therefore a = \sqrt{2578{\cdot}4} = 50{\cdot}78 \quad \therefore \text{chord} = 101{\cdot}6 \text{ cms.}$$

3. In a $\odot$, radius 5″, calculate lengths of chords at distances 1″, 2″, 2·5″, 3·78″ from the centre.

4. A bridge crosses a stream 35′ wide; its arch is an arc of a $\odot$ radius 50′: calculate to nearest inch how much the arch rises in the middle. Also make a careful sketch in which 10 feet is represented by 1 in., and check the calculation by measurement.

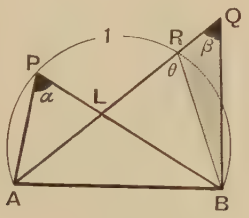
5. Prove that the middle points of chords of a $\odot$ equidistant from its centre lie on another $\odot$.

6. Prove that the two straight lines which join either directly or transversely the ends of two parallel chords of a $\odot$ are equal to one another.

7. If **CA**, **CB** are radii of a $\odot$ at right angles to each other, and **AM**, **BN** are drawn perpendicular to any diameter meeting it in **M** and **N**, prove that **AM** is equal to **CN**, and **BN** is equal to **CM**.

PROP. 43 (CONVERSE OF PROP. 41).

If the straight line joining two points subtend equal angles at two other points on the same side of it, then the four points are concyclic.



Let AB subtend equal ∠s α, β at P, Q.

Required to prove: A, P, Q, B concyclic.

One of AP, AQ makes the greater angle with AB. Let it be AP.

∴ AQ crosses PB; let it cross at L.

A circle can be described about △ APB; let it be 1.

L is a point on chord PB of 1; ∴ L is a point within ⊙ 1;
∴ AL produced cuts 1 at some point R.

Then either (1) R coincides with Q;
or (2) R is within LQ;
or (3) R is beyond Q.

If (2) were true,
θ (ext. to △ BQR) would be $> \beta$ (int. non-adjacent).
But $\alpha = \theta$; ∴ α would be $> \beta$

But it is not; ∴ (2) is not true.

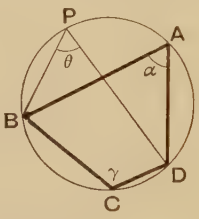
By a similar proof (3) is not true.

∴ (1) is true; *i.e.* A, P, Q, B are concyclic.

NOTE.—In the figure α, β are not actually equal, as is seen at once if a piece of paper cut to fit β be placed on α. The figure was made inaccurate in order to show the absurdity of supposing that R and Q do not coincide.

Prop. 44 (Converse of Prop. 42).

If the sum of a pair of opposite angles of a quadrilateral be equal to two right angles, the quadrilateral is cyclic, i.e. *a circle can be described passing through its corners.*



Let **ABCD** be the quadrilateral

Given: $\alpha + \gamma = 2$ rt. $\angle$s.

Required to prove: **ABCD** is cyclic.

Proof.—A circle can be described about **BCD.**

Let **P** be a point on it—not in the arc **BCD.**

Then **BPDC** is a cyclic quadrilateral

$\therefore \theta + \gamma = 2$ rt. $\angle$s (Prop. 42)

But $\alpha + \gamma = 2$ rt. $\angle$s (hyp.);

$\therefore \theta = \alpha$

$\therefore$ **B, P, A, D** are concyclic (Prop. 43)

i.e. **A** lies on the circle through **B, C, D**

i.e. **ABCD** is a cyclic quadrilateral.

Note.—Prop. 43 has an interesting application in navigation. Suppose **A** and **B** are two prominent objects—lighthouses or churches—on a coast, and that there are hidden rocks near **L.** An arc **1** is drawn on the chart, keeping well clear of the danger, and the value of α, called the "danger angle," is printed on the chart. A ship at **Q**, where the $\angle \beta$ subtended by **AB** is less than α, is in safety; hence when within possible range of danger the value of β is watched continually, and the ship's course is altered if β becomes nearly equal to α.

Exercise.

AOB is a triangle; **P** is a point in **AO**, **Q** a point in **BO**, such that $\angle$ **OQP** = $\angle$ **OAB**: show that the quadrilateral **ABQP** is cyclic.

PROP. 45. [Euclid III. 31.]

(1) *The angle in a semicircle is a right angle.*

(2) *The angle in a segment greater than a semicircle is less than a right angle.*

(3) *The angle in a segment less than a semicircle is greater than a right angle.*

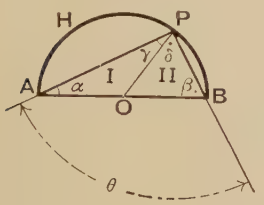


FIG. 1.

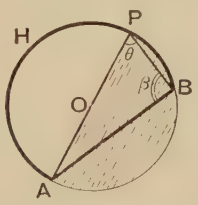


FIG. 2.

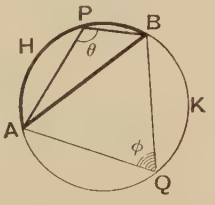


FIG.

Fig. 1. Let θ be $\angle$ at P in semicircle AHB.

Required to prove: θ is a rt. $\angle$.

Proof.—Draw radius OP.

Now, $\alpha + \beta + (\gamma + \delta) = 2$ rt. $\angle$s ($\angle$s of $\triangle$ APB)

But $\alpha = \gamma$, and $\beta = \delta$ ($\because$ I, II are isosceles);

$\therefore \gamma + \delta + (\gamma + \delta) = 2$ rt. $\angle$s

$\therefore \gamma + \delta =$ a rt. $\angle$

i.e. θ is a rt. $\angle$.

Fig. 2. Let θ be $\angle$ at P in seg. AHB, greater than a semicircle.

Required to prove: $\theta <$ rt. $\angle$.

Proof.—The magnitude of θ is the same wherever P may be on arc AHB, hence we may choose P, so that AP is a diameter.

Then $\theta + \beta < 2$ rt. $\angle$s (2 $\angle$s of $\triangle$).

But β is a rt. $\angle$ ($\angle$ in semicircle ABP)

$\therefore \theta <$ a rt. $\angle$.

Fig. 3. Let θ be $\angle$ at P in segment AHB, less than a semicircle.

Required to prove: $\theta >$ rt. $\angle$.

Proof.—Take $\angle$ ϕ at point Q in segment AKB.

APBQ is a cyclic quadrilateral; $\therefore \theta + \phi = 2$ rt. $\angle$s

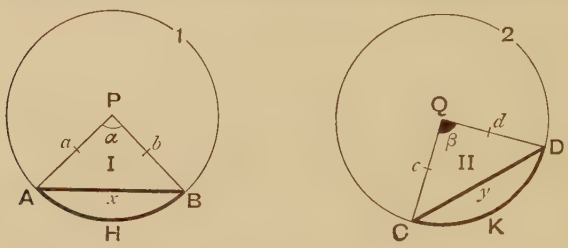
But $\phi <$ rt. $\angle$ ($\because$ segment AKB $>$ semicircle);

$\therefore \theta >$ rt. $\angle$.

*Prop. 46. [Euclid III. 26, 27, 28, 29.]

In equal circles, (1) *if two arcs subtend equal angles at the centre, then the arcs are equal;*

Converse, (2) *If the arcs be equal, then the angles are equal.*



Let AHB, CKD be arcs of equal ⊙s **1**, **2**.

Proof.—Place ⊙ **1** on ⊙ **2**, with centre P on centre Q; all radii of **1** = radii of **2**, ∴ **1** and **2** coincide.

Turn ⊙ **1** round the common centre till A lies on C.

Then

(1) **If** $\alpha = \beta$, PB lies along QD;

∴ B coincides with D,

and arc AHB coincides with arc CKD, *i.e.* arc AHB = arc CKD.

(2) **If arc** AHB = **arc** CKD, they must end at same point,

and ∴ B lies on D.

∴ PB coincides with QD,

and α „ „ β; ∴ $\alpha = \beta$

Corollary 1.—**If** $\alpha = \beta$, the sectors PAHB, QCKD are equal;

also **if arc** AHB = **arc** CKD, they are equal

(for in both cases they have been made to coincide).

Corollary 2.—∠s at circumference subtended by equal arcs are equal; (since ∠s at circumference = ½ ∠s at centre).

Corollary 3.—**If** $\alpha = \beta$, chord x = chord y;

also **if arc** AHB = **arc** CKD, they are equal

(for in both cases they have been made to coincide).

Corollary 4.—**If chord** x = **chord** y,

then △ I ≡ △ II (since $a = c$; $b = d$; and $x = y$)

∴ $\alpha = \beta$, and ∴ arc AHB = arc CKD

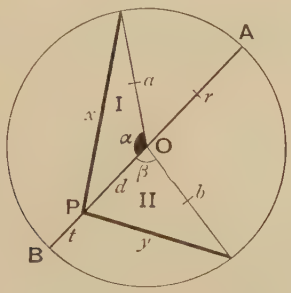
* Prop. 47. [Euclid III. 7.]

If from a point—not the centre—within a circle, straight lines be drawn to the circumference—

(1) *the greatest of these lines passes through the centre;*

(2) *the least is opposite to the greatest;*

(3) *of any two others, the one which subtends the greater angle at the centre is the greater.*



Let O be the centre of ⊙, P another point within it, and x, y lines drawn from P.

(1) **Required to prove:** PA $> x$.

I is a △; $\therefore d + a > x$

But $a = r$; $\therefore d + r > x$; *i.e.* PA $> x$.

(2) **Required to prove:** PB $< x$.

I is a △; $\therefore d + x > a$

But $a = d + t$; $\therefore d + x > d + t$

$\therefore x > t$; *i.e.* PB $< x$.

(3) **Given:** $\alpha > \beta$. **To prove:** $x > y$.

In △s I, II $\begin{cases} a = b \\ d = d \\ \alpha > \beta \end{cases}$

$\therefore x > y$

* Prop. 48. [Euclid III. 8.]

If straight lines be drawn to a circle from a point outside it—

(1) *the greatest is the one which passes through the centre;*

(2) *the shortest is the one which when produced passes through the centre;*

(3) *of any two others, the one which subtends the greater angle at the centre is the greater.*

This proposition resembles Prop. 47 so closely that the proof is left as an exercise.

Exercises.

1. Prove that a circle can be circumscribed about any quadrilateral whose sides are the bisectors of the angles of some other quadrilateral.

2. **P** is a point within a ⊙ whose centre is **O**: show that the greatest and least lines in the ⊙ which can be drawn through the point **P** are at right angles to one another.

3. Two opposite sides of a cyclic quadrilateral are ∥: prove that the other sides are equal.

4. Prove that the chord which bisects a radius of a ⊙ at right ∠s subtends an ∠ 60° at the circumference.

5. Prove that if the chord of half the arc of a circle is greater than the chord of the whole arc, the arc must be more than two-thirds of the whole circumference.

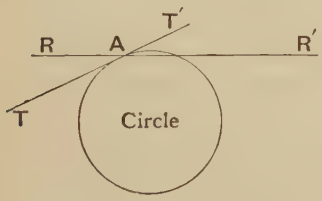
6. If the sum of two opposite angles of a quadrilateral is equal to two right angles, show that the perpendiculars drawn to its four sides at their middle points all meet in the same point.

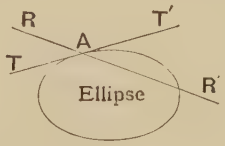
7. **ABCD** is a quadrilateral in a circle, and circles are described through **AD, BC**: if **BA, CD** produced meet the one in **K, L**, and **AB, DC** produced meet the other in **M, N**, prove that a circle will pass round **K, L, N, M.**

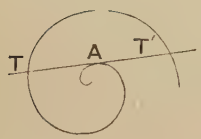
CHAPTER IX.

TANGENTS.

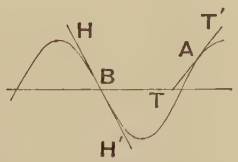
In each of the accompanying set of figures a line TAT′ is drawn which comes up to the curve at the point A, skims lightly along it, and then leaves it again. Such a straight







Spiral



Sine Curve

line **touches** the curve, and is hence called a **tangent**; the point A at which it touches is called the **point of contact.**

How can we distinguish between a line TT′ which touches a curve and one such as RR′ which clearly does not touch it?

In the case of the circle and the ellipse, we might say TT′ meets the curve at a single point A, whereas RR′ meets it at two points; but this test will not do for curves like the spiral or the sine curve, for in them TAT′, although a

tangent, does meet the curve again. Moreover, even in the case of the circle, the tangent and the curve very nearly coincide for a distance so appreciable as to make the point of contact very ill defined.

Try another test. Travel along the tangent from T to T′; as you approach A, the curve is on your right. You come up to the curve at A, and, passing on, the curve is still on your right; but travel along RAR′, as you pass A the curve crosses from right to left.

This test is satisfactory for the tangent TAT′ in all four figures. "If a line crosses the curve at a point it is not a tangent; if it does not cross it is a tangent." But this test also breaks down for the tangent HBH′ at B on the sine curve. At this point, which is called a Point of Inflection, the line which has been curving downwards bends back and begins to curve upwards; and the line HBH′, which certainly lies closer to the curve and touches it more decidedly than TAT′, crosses the curve at its point of contact.

Curves are Limiting Forms of Polygons.

We will now look at the matter in quite a different way. What *is* a circle? Something made by compasses? If so, fix your thoughts on the pencil point of your compasses, and try to think how it is moving while it marks, say, 0·001 centimetre along the circle. If you examined the mark with a good microscope you would probably not detect the difference between it and the same length of a straight line. Perhaps your thinking power is higher than the power of the microscope; if so, go on thinking of shorter and shorter lengths of the circle. In the long run you arrive at the same result, a line joining two points so close together that your mental microscope cannot distinguish between it and a piece of the straight line which runs through the points.

Or is the circle the locus of a set of points pricked off along radiating lines OP, OQ, OR, etc.? If so, in order to obtain a circle we must rule, say, a hundred equal lines

through O, and join up the end points by a line drawn "freehand." In doing this we should scarcely be conscious of departing from a straight line while passing from point to point. Carry the process further. Draw in imagination myriads of equal lines through O, join up the end points "freemind" instead of "freehand;" in the long run your mind fails to detect any difference between the piece of circle and the piece of straight line running from point to point.

The above discussion applies to any curve. Thus we have no simple compasses for drawing an ellipse, but we learn on p. 157 how to find individual points upon one. If such points be taken close enough together, we can detect no difference between the pieces of ellipse and the pieces of straight lines joining the points.

Hence we may think of a curve as the limiting form of some polygon with myriads of sides, each one shorter than any nameable length, however short.

Each side of the polygon may be called an **element** of the curve, and the ends of any side may be called **consecutive points** on the curve; the line formed by producing the side both ways is a tangent.

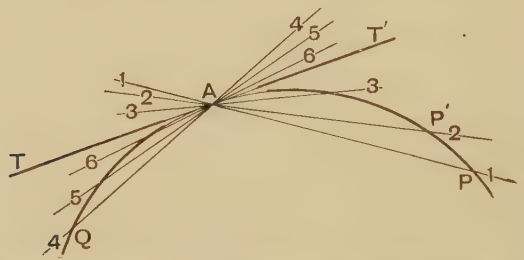
We have now reached a satisfactory definition.

Definition 1.—A **tangent** to a curve is a **straight line passing through two consecutive points** on the curve.

A Tangent the Limit of a Secant.

Again, draw the line **1** passing through the point **A** on the curve and the point **P** near to it. A line such as **1** is called a **secant.**

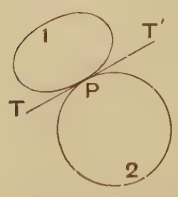
Now push **P** nearer to **A**, and draw the line **2**; push it nearer still, and draw the line **3**. Imagine **P** to move



right up to **A**; then **TAT'**, which is the limiting position taken up by the line as **P** arrives at **A**, is the tangent at **A**.

Definition 2.—A **tangent** to a curve at any point is the **limiting position of a secant** passing through the point when another point of intersection is made to move up to and to coincide with the first.

Definition.—Two curves are said to touch one another at a point if both pass through that point, and they have a common tangent at the point. Thus the curves **1, 2** touch at **P, TPT'** being the common tangent at **P**.



When **two circles** touch one another, they are said—

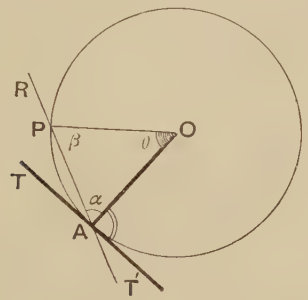
(*a*) to **touch internally**, if both lie on the **same side** of their **common tangent.**

(*b*) to **touch externally**, if they lie on **opposite sides** of their common tangent.

Prop. 49. [Euclid III. 16, 18, 19.]

The tangent at any point of a circle is at right angles to the radius drawn to that point.

Let A be any point on ⊙, centre O.
„ AT be the tangent to ⊙ at A.



Required to prove: AT ⊥ to AO.

Proof.—Draw APR through A and a neighbouring point P on the ⊙.

Draw radii OA, OP.

Then $\alpha + \beta + \theta = 2$ rt. ∠s (∠s of △ AOP)

Also $\beta = \alpha$ (∵ AO = OP)

∴ $2\alpha + \theta = 2$ rt. ∠s

Now move the point P gradually up to the point A. As P approaches A the line APR turns round A, and ultimately becomes the tangent AT; θ closes up and disappears at the instant when P reaches A.

Hence when P reaches A, $\theta = 0$; ∴ $2\alpha = 2$ rt. ∠s

∴ $\alpha =$ a rt. ∠

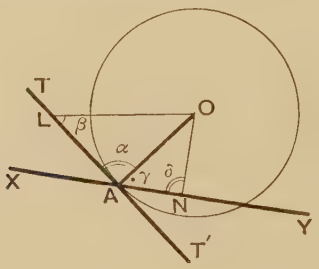
i.e. the tangent at A is at rt. ∠s to the radius.

Corollary 1.—There can be **only one tangent at the same point** on the circle (for only one line can be drawn at right angles to another at the same point).

Note.—Occasionally a curve has an abrupt change of direction at a point; it then has two tangents at the point, one corresponding to the direction of the curve on each side of the point.

Corollary 2.—The ⊥ drawn to any tangent from the centre passes through the point of contact (for the radius is such a ⊥, and only one ⊥ can be drawn from the same point).

Alternative Proof of Prop. 49.—(1) Draw TAT′ ⊥ to OA. Take any point L on AT; join OL.



$\alpha + \beta <$ 2 rt. ∠s (2 ∠s of △ AOL)
but $\alpha =$ rt. ∠; ∴ $\beta <$ a rt. ∠
∴ $\beta < \alpha$
∴ OA opposite to $\beta <$ OL opposite to α
i.e. OL > radius; *i.e.* L is outside the ⊙
Hence any point on TAT′, except A, is outside the ⊙;
∴ TAT′ meets the ⊙ at A, but does not cut it;
i.e. TAT′ is the tangent at A.

(2) Any other line through A must cut the ⊙.
Let XAY be a line through A not ⊥ to OA.
„ ON be ⊥ from O to XY.
Then $\delta =$ rt. ∠; ∴ $\gamma < \delta$ (as above)
∴ ON opposite to $\gamma <$ OA opposite to δ
i.e. ON < radius; *i.e.* N is within the ⊙
∴ at least one point of XAY is within the ⊙
∴ XY must cut the circle.

EXERCISES.

1. Prove that equal chords in a circle are tangents to a concentric circle.

2. ABC is an equilateral triangle inscribed in a circle : prove that the tangent at A is parallel to BC.

3. If a ⊙ touch each of two fixed straight lines, prove that its centre lies on one or other of two fixed straight lines.

4. Prove that the tangent at any point of a circle is parallel to the chords which are bisected by the radius drawn to the point of contact.

5. In a right-angled △ ABC, of which C is the right angle, CD is drawn perpendicular to the hypotenuse AB, meeting it in D : prove that the line CB is a tangent to the ⊙ described round the △ CDA.

6. A, B are points on an unlimited straight line ; tangents AP, BQ are drawn to *any* ⊙ touching the given line at any point : prove that either AP + BQ or AP − BQ = constant.

7. Tangents APX, BRY are drawn to a ⊙ at the ends of a diameter AB ; another tangent cuts AX at P and BY at R : prove that PR subtends a right angle at the centre of the ⊙.

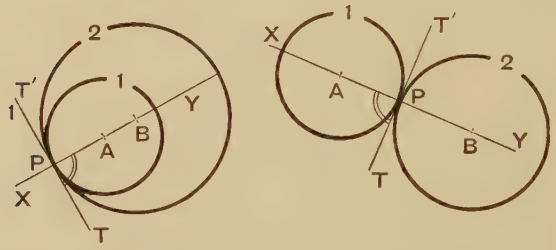
8. Prove that if a circle can be drawn to touch all four sides of a quadrilateral, the sum of one pair of opposite sides must be equal to the sum of the other pair.

* Prop. 50. [Euclid III. 11 and 12.]

If two circles touch one another, the straight line joining their centres passes through the point of contact.

Let **1**, **2** be the circles, having **TPT′** for common tangent at **P**.

Required to prove: **P** is on the line joining the centres.



Proof.—Through **P** draw **XPY** ⊥ to **TPT′**.

∴ **A**, the centre of **1**, lies on **XPY**
and **B**, „ „ **2**, „ „ } (Prop. 49)

∴ **XPY** passes through both **A** and **B**;

i.e. the straight line joining **AB** passes through **P**.

Note.—If **A** and **B** be the centres of two ⊙s, each is symmetrical about the line **AB**, for when produced it is a diameter of each ⊙; hence the ⊙s may be folded about the line **AB** so that their upper halves coincide with the lower halves.

If the circles intersect at **P** and **Q**, and **PN** be drawn ⊥ to **AB**, then **PN** folds into the position **QN**; hence the common chord **PQ** is ⊥ to **AB** and is bisected by it. The reader should draw a figure to illustrate this.

If the ⊙s meet at only one point **P**, the common chord **PQ** has become the common tangent **TPT′**; when folded, **TP** lies along **T′P**, which proves that the line of centres is ⊥ to the common tangent and passes through the point of contact.

Exercises.

1. Each of four unequal circles touches externally two and only two of the remaining three: prove that a circle can be drawn through the four points of contact.

2. **AB** is a line of 5 in.; from **A** as centre circles are described with radii 2 and 3 in., and from **B** as centre circles are described with radii 2, 3, and 4 in.: point out the circles which do not cut, which do cut, and which touch one another, stating your reasons.

3. From the point **T** external to the circle **QPR**, whose centre is **O**, the lines **TP** and **TRQ** are drawn touching the circle at **P** and cutting it in **Q** and **R** respectively, and **QR** is bisected at **S**: prove that—

(1) The angles **PST** and **POT** are equal.

(2) If **PS** produced meets the circle again in **U**, and **UV** drawn parallel to **QR** meets the circle again in **V**, then **TV** touches the circle.

4. The line **ACD** touches the circle **AEF** at **A**, and passes through the centre **G** of the circle **BCD**; and the line **BEF** touches the circle **BCD** at **B**, and passes through the centre **H** of the circle **AEF**: prove that $\angle$ **AHB** = $\angle$ **BGC** or $\angle$ **BGD**.

5. **ABC** is an isosceles triangle; from the equal sides **AB**, **AC** equal parts **AX**, **AY** are cut off; a circle is drawn through **B** and **X**, and cuts **XC**, **BY** produced at **E** and **F**: prove that **EF** is parallel to **AC**.

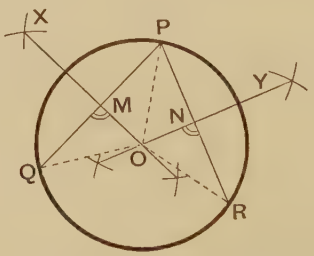
6. Two circles **ABC**, **BCD** meet in **BC**: if **AB**, **AC** be tangents to the circle **BCD**, prove that the centre of **BCD** lies on the circumference of **ABC**.

CHAPTER X.

CONSTRUCTIONS CONCERNING CIRCLES.

CONSTRUCTION 15. [**Euclid III. 1.**]

Being given a circle whose centre is not marked, required to find the centre.



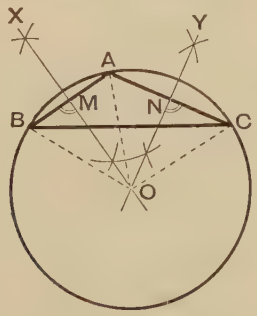
Through any point P on given ⊙ draw two chords PQ, QR.
Draw MX, bisecting PQ at right ∠s.
„ NY, „ PR „ „
MX, NY intersect at O, which is required centre.

Proof.—PQ, PR are not ∥ ; ∴ MX, NY, which are ⊥ to them, are not ∥, but must meet at some point O.

All points equidistant from P and Q lie on MX ;
the required centre is equidistant from P and Q.
∴ centre must lie on MX
Similarly „ „ NY } ∴ O must be centre

CONSTRUCTION 16. [Euclid IV. 5.]

Required to describe a circle about a given triangle; i.e. *to draw a circle passing through its angular points.*



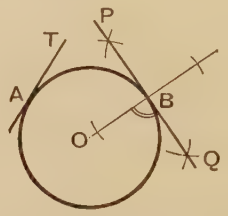
Let ABC be the given △
Draw MX, bisecting AB at right ∠s.
„ NY, „ AC „ „
MX, NY intersect at O, which is required centre.

Proof.—Same as last proposition.

NOTE.—In this figure ∠ at A is obtuse; the student should draw figures for △s having ∠ at A (1) a right ∠, (2) acute, and notice the change in position of O.

Construction 17.

Required to draw the tangent at a given point on a given circle.



First Method.—To draw the tangent to ⊙ at **A**. Slide a straight-edge over the ⊙ till it touches it at **A**, as well as the eye can judge; then rule the tangent **AT**.

Second Method.—To draw the tangent at **B**.

Draw the radius **OB**, and produce it.

Construct **PBQ** ⊥ to **OB** at **B**.

PBQ is the tangent at **B** (for tangent is at rt. ∠s to radius).

Exercises.

In the figure on p. 145, let **d** = **OA** = distance of **A** from centre; **t** = **AP** = length of tangent from **A** to circle; **r** = radius;

$$\therefore\ d^2 = t^2 + r^2$$

1. Given $r = 2{\cdot}3''$, $d = 3{\cdot}9''$, calculate **t**.

$t^2 = 3{\cdot}9^2 - 2{\cdot}3^2 = 6{\cdot}2 \times 1{\cdot}6 = 9{\cdot}92 \quad \therefore t = \sqrt{9{\cdot}92} = 3{\cdot}15''$

2. Given $r = 3''$, $d = 5''$, calculate **t**.
3. „ $r = 12{\cdot}8$ cms., $d = 21{\cdot}3$ cms., calculate **t**.
4. „ $r = 6$ cms., $t = 11$ cms., calculate **d**.
5. „ $d = 4{\cdot}1''$, $t = 2{\cdot}55''$, calculate **r**.

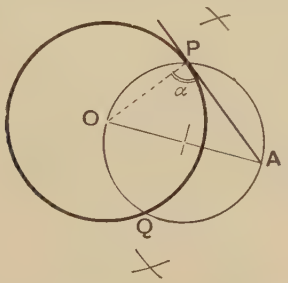
6. A car, which has been moving at 15 miles per hour round a circular track of radius 470 feet, goes off at the same pace along a tangent: how far will it be from the centre after 5 sec.; and also after half a minute?

7. A man is in a balloon at a height 4 miles above a level country: how far ought he to see the surface of the ground in every direction, supposing the Earth to be a sphere of 4000 miles radius?

8. Draw a careful figure for each of the above examples, and check your calculation by measurements.

CONSTRUCTION 18. [Euclid III. 17.]

Required to draw a tangent to a circle from a point outside it.



Let **O** be centre of given ⊙, **A** the point from which a tangent is to be drawn.

First Method.—Join **OA**, and bisect it.

Describe ⊙ on **OA** as diameter, cutting given ⊙ at **P** and **Q**.

Join **AP**; it is the required tangent.

If **AQ** were joined, it also would be a tangent.

Proof.— α = a rt. ∠ (∠ in semicircle);

∴ **AP**, being ⊥ to radius **OP**, is a tangent.

Second Method.—Place a straight-edge so as to pass through **A** and touch the ⊙, as well as the eye can judge; then rule the tangent.

NOTE.—This method gives quite as true a tangent as the first method, but it does not fix the point of contact **P**, and this is difficult to judge by eye.

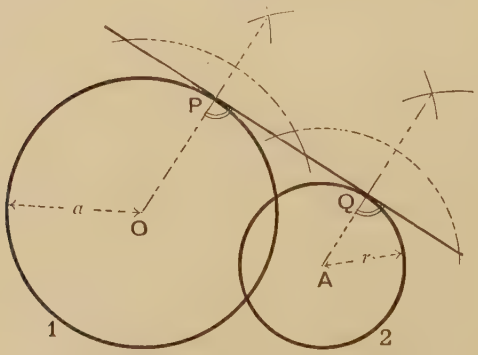
Definition.—A straight line which touches each of two curves is called a common tangent to them.

A common tangent to two circles is called **direct** if its points of contact are on the same side of the line joining the centres; it is called **transverse** when they lie on opposite sides of this line.

CONSTRUCTION 19.

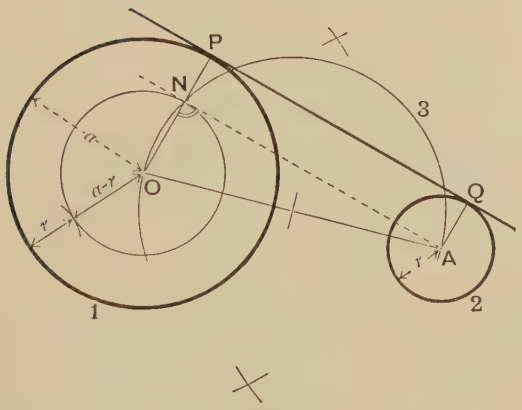
Required to draw a direct common tangent to two given circles.

Let O be the centre, a the radius of ⊙ 1.
A „ „ r „ „ 2.



First Method.—Slide a straight-edge over the ⊙s till it touches both; then rule the common tangent.

The points of contact P and Q can then be found by constructing OP, AQ, ⊥s to the tangent from O and A.



Second Method.—Join OA; bisect it.

Describe ⊙ 3 on OA as diameter.

From the outer end of a radius of 1 cut off a length r.

Describe ⊙, centre O, radius $a-r$, cutting 3 at N.

In ⊙ 1 draw radius ONP passing through N;

„ 2 „ AQ, || to OP (by means of set squares).

Join PQ and produce it. PQ is the required direct tangent.

Proof.—$\left.\begin{matrix} OP = a \\ ON = a-r \end{matrix}\right\} \therefore PN = r = AQ$

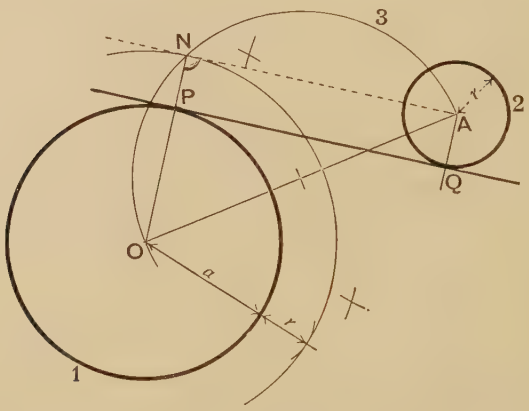
Also PN, AQ are ‖; ∴ AQPN is a ▱.

∠ ONA is a rt. ∠ (∠ in semicircle); ∴ AQPN is a rectangle.

∴ PQ is ⊥ to radii OP, AQ, and touches both 1 and 2.

Construction 20.

Required to draw a transverse common tangent to two given circles.



Let O be the centre, a the radius of ⊙ 1.

A „ „ r „ „ 2.

First Method.—Same as for direct tangent.

Second Method.—Join OA; bisect it.

Describe ⊙ 3 on OA as diameter.

Produce a radius of 1, and cut off a length r.

Describe ⊙ centre O, radius $a + r$, cutting 3 at N.

Join ON, cutting 1 at P.

In ⊙ 2 draw radius AQ, || to OP (by means of set-squares).

Join PQ and produce it. PQ is the required transverse tangent.

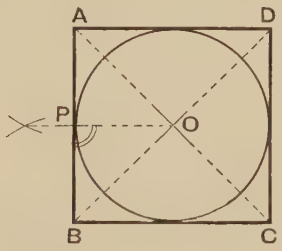
Proof.—Similar to proof or external tangent.

Note 1.—If the ⊙s intersect, they have no transverse common tangent.

Note 2.—Constructions 19 and 20 easily give two direct and two transverse common tangents.

CONSTRUCTION 21. [Euclid IV. 8.]

To inscribe a circle in a given square.

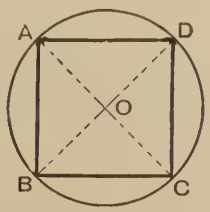


Let **ABCD** be the square.
Draw **AC, BD** cutting at **O**.
From **O** draw **OP** $\perp$ to **AB**.
Describe $\odot$ centre **O**, radius **OP**.
It is the required circle.

Proof.—An easy exercise.

CONSTRUCTION 22. [Euclid IV. 9.]

To describe a circle about a given square.

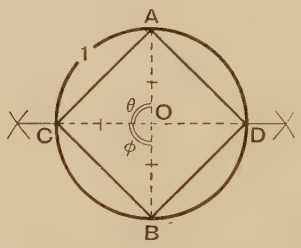


Let **ABCD** be the square.
Draw **AC, BD**, cutting at **O**.
Describe $\odot$ centre **O**, radius **OA**.
It is the required $\odot$.

Proof.—An easy exercise.

Construction 23. [Euclid IV. 6.]

To inscribe a square in a given circle.

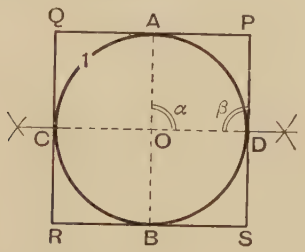


Let **1** be the given ⊙.
Draw any diameter **AOB**.
„ diameter **COD** ⊥ to it.
Join **AC, BC, AD, BD.**
ACBD is the required square.

Proof.—Each ∠ of **ACBD** is a rt. ∠ (∠ s in semicircle).
Also **AO** = **BO** (radii)
CO = **CO**
$\theta = \phi$ (rt. ∠ s)
$\therefore \triangle AOC \equiv \triangle BOC$
$\therefore$ **AC** = **BC**
Similarly all the sides are equal ;
$\therefore$ **ACBD** is a square.

CONSTRUCTION 24. [Euclid IV. 7.]

To describe a square about a circle.



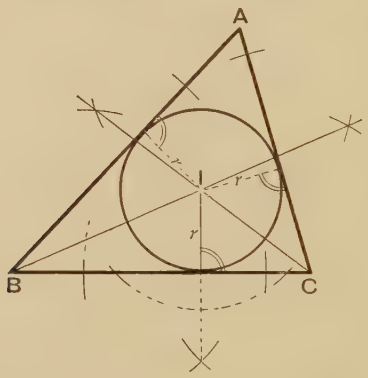
Let **1** be the given ⊙.
Draw any diameter **AOB**.
„ diameter **COD** ⊥ to it.
Through **A, B** draw **PQ, RS** || to **CD**.
„ **C, D** „ **QR, PS** || to **AB**.
PQRS is the required square.

Proof.—**OP** is a ▱, α is a rt. ∠, **OA** = **OD** (radii);
∴ **OP** is a square; so **OQ, OR, OS** are squares;
∴ **PQRS** is a square.

Again, β is rt. ∠ (∵ **OP** is square);
∴ **PS** touches ⊙;
so **QR, RS, QP** touch ⊙.

Construction 25. [Euclid IV. 4.]

To inscribe a circle in a given triangle, that is, to describe a circle within the triangle, touching each of its sides.



Let ABC be the given △.
Bisect ∠s at B and C by straight lines cutting at I.
From I draw r ⊥ to BC.
Describe ⊙ centre I, radius r; it is required ⊙.

Proof.—I is on bisector of ∠ ABC;
∴ ⊥ from I on AB = ⊥ from I on BC
I is on bisector of ∠ ACB;
∴ ⊥ from I on AC = ⊥ from I on BC
∴ each of the three ⊥s = r
∴ ⊙ centre I, radius r, passes through foot of each ⊥.
Also each side touches ⊙, since tangent is ⊥ to radius.

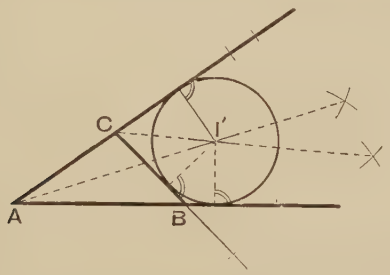
Definition.—This ⊙ is often called the **incircle**, and its centre the **incentre**.

Exercise.

The inscribed circle of a △ ABC touches BC at F: prove that AB + FC = AC + FB.

CONSTRUCTION 26.

To draw a circle outside a given triangle, touching one of the sides and the other two sides produced.



Let ABC be the given △.

At A bisect the internal ∠ of △.

C „ external ∠ of △.

Let bisectors intersect at I'.

From I' draw a ⊥ to BC (set-square).

With this ⊥ as radius, describe ⊙ centre I'; it is the required ⊙.

Proof.—Similar to proof of Construction 25.

Definition.—A ⊙ touching one side of a △ and two other sides produced is called an **escribed** ⊙, or an **excircle**, and its centre is called an **excentre.**

Every △ has three escribed ⊙s.

Exercises.

1. From a point **P** outside a circle (2 in. radius) draw a line **PQR** such that the part **QR** within the circle is 3 in. long.

2. Describe a circle passing through a given point and touching a given circle at a given point.

3. Show that if the centres of the inscribed and circumscribed circles of a triangle coincide, the triangle must be equilateral.

4. If a number of triangles have one side and the centre of the circumscribed circle common to all, prove that the centres of the inscribed circles lie on the circumference of one or other of two fixed circles.

5. The inscribed circle of a triangle **ABC** touches the sides **AB** and **AC** at **N** and **M**; a concentric circle cuts **AM** at **P**, **AN** at **Q**: show that **PM** = **QN**.

6. Find the centre of a circle which cuts off equal chords from the sides of a given triangle.

7. Prove that the tangents drawn to the inscribed circle of a triangle **ABC**, at points which are situated on a certain part of its circumference, form, with the lines **AB** and **AC**, triangles whose perimeters are all equal.

8. Describe a $\odot$ equal to a given $\odot$ so as to pass through a given point, and touch a given straight line. Under what circumstances is this problem impossible?

9. Triangles are drawn on the same side of a given base and have equal vertical $\angle$s: prove that the bisectors of the vertical $\angle$s all pass through a fixed point.

10. **AP** is a diameter of a circle circumscribing the equilateral triangle **ABC**, and **PC** is joined: if **O** be the centre of the circle, show that **OCP** is an equilateral triangle, and that the square on **AC** is three times the square on **OA**.

11. Through a given point within a circle of 3 cms. radius draw a chord of length 5 cms.

12. Describe a $\odot$ to touch each of two given lines and having its centre at a given distance from a third given line: how many such circles can be described?

CHAPTER XI.

LOCI AND HINTS TOWARDS WORKING EXERCISES.

Loci.

If any one were told to mark on a sheet of paper a point "anywhere he pleased," his choice would then be absolutely unrestricted and free from any "condition."

But suppose, on the other hand, he were told to mark the point exactly 2 cms. from a corner; this would be imposing a condition. The choice between an endless number of points would still be open to him; yet the choice would be restricted by the **condition** which the point must **satisfy.**

The **complete collection** of **suitable positions** we call the **locus** (*i.e.* the place) **of points satisfying the condition.**

If a quarter-circle, radius 2 cms., and centre at a corner, were drawn, would that be the locus? Every point on the quarter-circle satisfies the condition of being 2 cms. from a corner—hence the quarter-circle is certainly *part* of the locus; but it omits the points which are 2 cms. distant from the other corners.

The complete locus is four quarter-circles, one with a centre at each corner.

From the above example, we see that, in order to prove that any line, or group of lines, is the locus of a point, which satisfies a given condition, we must prove that—

(1) every point on the lines satisfies the condition,

(2) no other points satisfy it.

EXAMPLE 1.—What is the locus of a point distant 5 cms. from a fixed point **O**? A circle, centre at **O**, radius 5 cms.

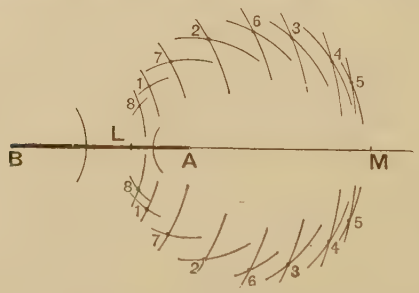
For (1) Every point on the circle is 5 cms. from **O**.

(2) No point is 5 cms. from **O** unless it be on the circle.

EXAMPLE 2.—What is the locus of a point distant 2 in. from a fixed straight line **XY**? A pair of straight lines ‖ to **XY**, one on each side of it at a distance 2 in.

EXAMPLE 3.—What is the locus of a point **P** which moves so that a fixed line **AB** always subtends the same ∠ at **P**? Two circular arcs, one on each side of **AB**, and terminated at **A** and **B**. (Props. 41 and 43.)

EXAMPLE 4.—**AB** is a fixed line 2·4 cms. long: what is the locus of a point **P** which moves so that its distance from **B** is always double its distance from **A**; *i.e.* so that **BP** = 2**AP**?



Here we cannot, as in the last three examples, state right away what the locus is; hence we proceed experimentally.

May we have **AP** = 1 cm.? Yes, provided **BP** = 2 cms.

Describe ⊙ centre **A**, rad. 1 cm., and ⊙ centre **B**, rad. 2 cms.;
These ⊙s intersect at points **1, 1**, which are on the locus.

May we have **AP** = 1·5 cm.? Yes, provided **BP** = 3 cms.

Describe ⊙ centre **A**, rad. 1·5 cm., and ⊙ centre **B**, rad. 3 cms.;
These ⊙s intersect at points **2, 2**, which are on the locus.

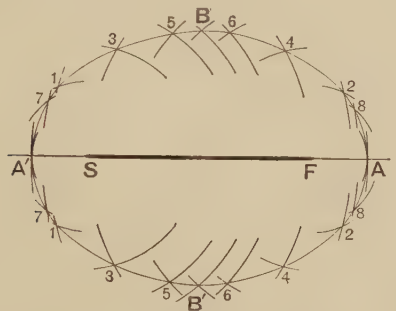
Similarly we find the points **3, 3** (radii 2 cms., 4 cms.);
4, 4 (radii 2·2, 4·4); **5, 5** (radii 2·3, 4·6); **6, 6** (radii 1·8, 3·6);
7, 7 (radii 1·2, 2·4); **8, 8** (radii 0·9, 1·8).

⊙s, radii 2·4, 4·8, touch at **M** and give only one point on locus.
⊙s, „ 0·8, 1·6, „ **L** „ „ „ „

The ⊙s for values of **AP** less than 0·8 cm. or greater than 2·4 do not intersect; hence such values of **AP** are not admissible.

It is now fairly clear from the figure that the required locus is a circle on **LM** as diameter. This result, arrived at experimentally, cannot be proved until Section II. is read.

EXAMPLE 5.—**SF** is a fixed line 3 cms. long: find the locus of a point **P** which moves so that **SP** + **FP** = 4·5 cms.



Here also we must proceed experimentally.

May we have **SP** = 1 cm.? Yes, provided **FP** = 3·5 cms.

Describe ⊙ centre **S**, rad. 1 cm., and ⊙ centre **F**, rad. 3·5 cms. These ⊙s intersect at **1, 1**, which are on the locus.

Similarly we find the points—

2, 2 (radii 3·5, 1)
3, 3 (radii 1·5, 3)
5, 5 (radii 2, 2·5)
7, 7 (radii 0·9, 3·6)
A (radii 0·75, 3·75)
B, B (radii 2·25, 2·25)
4, 4 (radii 3, 1·5)
6, 6 (radii 2·5, 2)
8, 8 (radii 3·6, 0·9)
A' (radii 3·75, 0·75)

Having constructed these points, the curve drawn freehand through them is the required locus; it is called an **ellipse.**

NOTE.—A fairly accurate ellipse may be drawn as follows: stick pins firmly into the paper at **S** and **F**, leaning slightly outwards towards **A** and **A'**; tie a piece of thread into a loop, place the loop loosely over the pins, insert the point of a pencil in the loop, and, keeping the loop extended with it, run the pencil along the paper round the curve, allowing the thread to guide the point.

Exercises.

1. What is the locus of the vertex of isosceles triangles having a given base? [First, try experimentally. Take a fixed line **BC**, say 1·5″ long; on it construct isosceles △s with their equal sides 1″, 1·5″, 2″, etc. When you can detect the locus in your figure, prove that your supposed locus is correct.]

2. (Important.) Prove that the locus of points which are equidistant from two intersecting lines **AOB, COD** is the pair of lines which bisect the ∠s **AOC, AOD.**

3. (Important.) Find the locus of points which are equidistant from two fixed points.

4. What is the locus of points equidistant from two given ‖ lines?

5. **a, b** are given ‖ lines: what is the locus of a point **P**, whose ⊥ distance from **a** = twice its ⊥ distance from **b**?

6. What is the locus of the vertex of △s having a given base and given area?

7. **PL** is the ⊥ drawn from **P** to a fixed line **OX**;
PM „ „ „ **P** „ „ **OY**;
find the locus of **P**: (1) when **PL** = 2**PM**; (2) when **PL** + **PM** = a fixed length (say 4 cms.).

8. The hypotenuse **BC** of a right-angled △ **PBC** is fixed: what is the locus of **P**?

9. What is the locus of the centre of a ⊙ of given radius (say ¾″) which touches a given ⊙ (say radius 1½″) externally? Also what is the locus for internal contact?

10. **ABC** is a △. In **AC** a point **E** is taken, and in **AB** a point **F**; the ▱ **AEHF** is completed, and **BE, CF** are joined: prove that if **E, F** move along the sides of the △ in such a way that the sum of the areas of △s **AEB, AFC** remains always the same, **H** moves along a straight line ‖ to **BC**.

11. From the fixed points **A, B** of a ⊙, ‖ straight lines **AP, BQ** are drawn to cut the ⊙ again in **P** and **Q**: prove that the locus of middle point of **PQ** is a concentric circle.

12. Any point **P** is taken on the arc of a segment of a ⊙ of which **AB** is the chord, and the lines **AD, BE**, drawn perpendicular to **BP, AP** respectively, meet in the point **Q**: show that locus of **Q** is a circle through **A** and **B**.

13. Through one of the points of intersection of two circles, centres **A** and **B**, a chord is drawn meeting the circles at **P** and **Q** respectively; the lines **PA, QB** intersect in **C**: find the locus of **C**.

14. From a point **P** in the base **BC** of an isosceles triangle **ABC**, perpendiculars **PL**, **PM** are drawn to the sides **AB**, **AC**: prove that the sum of **PL** and **PM** is the same for all positions o **P**. How is this theorem modified when **P** is in **BC** produced?

15. Prove that the locus of a point such that the difference of the squares on its distances from two given points is equal to a given square is a pair of straight lines.

16. If **P** be any point inside a circle, prove that the middle points of all the chords drawn through **P** are situated on another circle.

17. What is the locus of points from which tangents of a given length can be drawn to a given $\odot$?

18. Two straight bars **AX**, **BY** rotate round fixed points **A**, **B** with equal speed: show that when they rotate in the same direction their point of intersection describes a circle.

19. The base of a $\triangle$ is fixed, its vertical $\angle$ is given in magnitude: find locus of its incentre.

20. The circumference of a $\odot$ of given radius always bisects the circumference of a fixed $\odot$: find the locus of the centre of the variable $\odot$.

21. Draw a line **AB** of length 8 cms.: find experimentally the locus of a point **P** which moves so that $AP = 3BP$.

22. Draw a line **AB** of length 3″: find experimentally the locus of **P**, if $BP = 4AP$.

23. Draw a line **SF** of length 8 cms.: find experimentally the locus of a point **P**, which moves so that $SP + FP = 10$ cms.

24. Take $SF = 2{\cdot}5''$: find experimentally the locus of **P**, if $SP + FP = 4''$.

Hints towards working Exercises.

A.—Problems.

(1) Think over the constructions given in the book. See whether any of them resemble the required construction; if so, try a similar method.

(2) Draw a careful figure showing any given lines, circles, triangles, etc. On this, sketch the required lines, judging their proper position as well as you can by eye. Next, assume for the moment that your sketch is correct, and see what consequences follow from it; *e.g.* see what lines or angles are equal, which were neither given equal nor required to be equal. This process is the **analysis** of the problem.

Then try to work backward, and to build up the required construction by the help of the results obtained. This process is **synthesis**.

Example 1.—Given a $\triangle$ ABC, required to draw a line PQ $\parallel$ to BC, meeting AB at P, and AC at Q, and such that BP = BQ.

Analysis.—Draw a convenient $\triangle$, making, say, BC = 6 cms., CA = 5 cms., AB = 8 cms. Place a straight-edge at A $\parallel$ to BC, slide it downward until apparently BP = PQ. Join BQ. At once you notice $\triangle$ PBQ is isosceles; hence $\angle$ PBQ = $\angle$ PQB. Also PQ being $\parallel$ to BC, BQ makes equal alternate $\angle$s with it. Hence BQ bisects $\angle$ ABC.

Synthesis.—Draw BQ bisecting $\angle$ ABC; let it meet AC at Q. Through Q draw QP $\parallel$ to BC. It is the required line.

Proof:—Easy.

EXAMPLE 2.—Required to draw through a given point **P** in the base **BC** of a given △ **ABC**, a line which bisects the △.

Analysis.—Draw a convenient △, making, say, **BC** = 8 cms., **CA** = 5 cms., **AB** = 9 cms.; take **BP** = 3 cms. Place a straight-edge passing through **P**, turn it round **P** until you think it makes a △ **PQC** = $\frac{1}{2}$ △ **ABC**. This is difficult to judge. To help your eye, join **L**, the middle point of **BC**, to **A**; then △ **ALC** = $\frac{1}{2}$ △ **ABC**. ∴ △ **PQC** is required equal to △ **ALC**. If it be so, join **QL**; then △ **CLQ** may be taken from each;

∴ △ **ALQ** = △ **PLQ**. ∴ **AP**, **QL** are ∥.

Synthesis.—Given △ **ABC** and point **P** in **BC**. Bisect **BC** at **L**. Join **AP**. Draw **LQ** ∥ to **AP**, cutting **AC** at **Q**. Join **PQ**; it is the required line.

Proof. –△ **PLQ** = △ **ALQ** (same "base" and same "height").

Add △ **CLQ** to each; ∴ △ **PQC** = △ **ALC** = $\frac{1}{2}$ △ **ABC**.

NOTE.—If **P** were nearer to **C** than to **B**, the wording of both analysis and synthesis would need a slight modification.

EXERCISES.

1. **BAC** is half a right angle. Through any given point show how to draw a straight line making with **AB** an angle equal to twice the angle it makes with **AC**.

2. Through a given point draw a straight line to cut two parallel straight lines, so that the part of it intercepted by them may be equal to a given straight line. Discuss the cases when two, one, or no straight lines can be drawn.

3. Through a given point on a side of a parallelogram draw a line so as to bisect its area.

4. The side **AB** of a triangle **ABC** is produced to **X**: find a point **Y** on **AC** or **AC** produced, such that the triangle **AXY** is double the triangle **ABC**.

5. Divide a straight line into two parts, such that the square on one part is double that on the other part.

6. Draw a straight line parallel to a given straight line, so that the part intercepted between two given straight lines may be of given length.

7. **AB**, **AC** are two given straight lines: show how to draw through **B** a line **BPQ**, cutting **AC** at **P**, and such that if produced to **Q** so that **PQ** = **PA**, then ∠ **QCA** = ∠ **QBA**.

(3) **Intersection of Loci.**—Frequently it is required to find a point which satisfies two distinct conditions; say, it is to be at a given distance from some line, and also to be equidistant from two given points; or, say, it is to be equidistant from two given lines, and also equidistant from two given points.

In such problems, first find the locus of all points which satisfy the first condition, then the locus of all which satisfy the second. Then—

(*a*) If the loci intersect only at one point, that is the required point.

(*b*) If the loci intersect in several points, any one of them will serve as the required point.

(*c*) If the loci do not intersect, there is no solution of the problem.

EXAMPLE 1.—Find a point which is 8 cms. from a given point **A** and 3 cms. from a given line **k**.

The locus of all points 8 cms. from **A** is a ⊙, centre **A**, radius 8 cms.

The locus of all points 3 cms. from **k** is a pair of straight lines ‖ to **k** and 3 cms. distant from it on opposite sides.

According to the position of **A**, the result is as follows :—

The ⊙ may cut both lines, giving 4 points as solutions.
" " touch both lines, giving 2 points as solutions.
" " cut only one line, giving 2 points as solutions.
" " touch one line, giving 1 point as solution.
" " cut one and touch the other, giving three points as solutions.
" " meet neither, giving no point as solution.

The student should draw figures for each case.

EXAMPLE 2.—Construct a circle of given radius (*e.g.* 2 cms.) touching externally two given circles (*e.g.* radii 1·5 cm., 3 cms.).

The locus of the centre of ⊙ radius 2 cms. touching the ⊙ radius 1·5 cm. is a concentric ⊙ radius 3·5 cms. (2 + 1·5).

The locus of the centre of ⊙ radius 2 cms. touching the ⊙ radius 3 cms. is a concentric ⊙ radius 5 cms. (2 + 3).

The points of intersection—if there be any—of these two loci are possible centres of the required ⊙.

B.—THEOREMS.

(1) Draw an accurate figure, and place corresponding marks on all lines, angles, or areas which are given equal in the hypothesis. In drawing the figure, take care as far as possible not to make any lines or angles equal, or any angles right angles, etc., unless either they are so given in the hypothesis, or unless it can readily be seen that they must be so. For example, do not make triangles isosceles unless so given; do not draw rectangles when parallelograms are given; if there be two circles in the figure, make them with different radii, unless given equal, and so on.

(2) Compare with any similar proposition given in the book, and try the method there used.

(3) Assume for the moment that the theorem is true, and examine what consequences follow from the assumption. You will then probably find that the theorem is true provided certain lines, or areas, or angles are equal; and the proof of such equality may be easy.

(4) Try the *reductio ad absurdum*, *i.e.* assume for the moment that the proposition is not true, and try to prove that some erroneous or absurd consequence follows from this assumption.

This method is used in Props. 7 and 43.

(5) Try the method of exhaustion, *i.e.* make a full list of all possible results that might follow from the hypothesis; then prove one by one that none of them are true except the required result.

This method is used in Props. 17 and 20.

Miscellaneous Exercises.—I.

1. Prove that any two straight lines drawn through the centre of a square at right angles to each other and cutting the sides divide the square into four equal parts.

2. **ABC** is a triangle, **BD**, **CE** lines making equal angles with **BC**, and meeting the opposite sides in **D** and **E**, and each other in **F**: prove that, if the angle **AFE** is equal to the angle **AFD**, the triangle is isosceles.

3. **ABCD** is a quadrilateral having the side **AB** parallel to the side **DC**, and **E** is the middle point of the side **AD**; through **E** a line is drawn parallel to **AB**: prove that if any point **P** be taken in that line, then the triangles **PAB**, **PCD** taken together are half the quadrilateral.

4. Show that every triangle which is not equilateral must have one angle greater than the angle of an equilateral triangle.

5. Through the middle point of the base of a triangle a straight line is drawn equally inclined to the sides: prove that the segment measured from the vertex, which it intercepts on either side, is equal to half the sum of the two sides of the triangle.

6. **P** is a point in the hypotenuse **BC** of a right-angled $\triangle$ **ABC**: if $\angle$ **BPA** $= 2\ \angle$ **BCA**, prove that $\angle$ **CPA** $= 2\ \angle$ **CBA**.

7. Find the point the sum of whose distances from the four angular points of a quadrilateral is least.

8. If **BDEC** be the square described externally on the side **BC** opposite to the right angle **BAC** of a right-angled triangle **ABC**, and **AD** be joined, prove that the angles **BAD** and **BDA** are together equal to the angle **BCA**.

9. Prove that the sum of the distances of any point from the angular points of a polygon is greater than half the perimeter.

10. Prove that the two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle of the base.

11. If one triangle lie entirely inside another one, prove that the perimeter of the former is less than the perimeter of the latter. Is a corresponding theorem true for two quadrilaterals?

12. Four straight lines meet at a point so that the opposite angles are equal: prove that the straight lines are two and two in the same straight line.

13. Show that the perimeter of a polygon is greater than the perimeter of any polygon lying wholly inside it, provided the latter is not re-entrant.

MISCELLANEOUS EXERCISES.—II.

1. **ABC** is a △ having **AB** = **AC**: find points **L** in **AB** and **M** in **AC** such that **LM** = **BL** = **CM**.

2. If **AB, CD** are equal straight lines given in position in a plane, find a point **P** in another given straight line such that △ **PAB** = △ **PCD**. Is a solution of the problem always possible?

3. From a given point in the base of a △ draw two straight lines so as to trisect the △.

4. **AB** is a given straight line, **P** is a point not upon it: find two points **Q, R** in **AB** or **AB** produced such that △ **PQR** is equilateral.

5. Having given the base and the circumcentre of a triangle, prove that the bisector of the vertical angle always passes through one of two fixed points.

6. Determine all the points, each of which is equidistant from three given straight lines.

7. Find a point equidistant from two of the angular points of a △ and at a given distance from the third.

8. Prove that the straight line bisecting an ∠ of a quadrilateral inscribed in a ⊙ meets the external bisector of the opposite ∠ on the ⊙.

9. Through a given point draw two straight lines containing a given angle and intercepting a given length on a given straight line. When is it impossible to do this?

10. Describe a ⊙ to touch two given lines and pass through a given point.

11. Draw when possible a ⊙ of given radius through a given point to touch a given ⊙.

12. Construct a right-angled triangle of which the hypotenuse is given in length and position, while the opposite vertex lies on a given straight line. How many solutions are there to this problem?

13. **AB, CD** are two ∥ straight lines, and **EF** is a line cutting them: draw two circles, one on each side of **EF**, touching **EF** and each of the two straight lines.

14. **AB, CD** are two given straight lines, and **X** a given point in **AB**: find a point **Y** in **AB** such that **YX** may equal the perpendicular distance of **Y** from **CD**.

15. Draw through a given point **P** in the side **AB** of △ **ABC** a straight line **PQ** cutting **AC** or **AC** produced in **Q** such that the area of △ **APQ** = half the area of △ **ABC**.

16. Two ⊙s with centres A and B touch each other externally at P; QR is a common tangent: prove that QR is also a tangent to the ⊙ described on AB as diameter.

17. If two unequal circles cut one another, how must they be drawn so that the common chord shall be the greatest possible?

18. Find a point such that the tangents drawn from it to two given circles shall be equal to two given straight lines:

19. The sides of a ▱ touch a ⊙: prove that the ▱ must be a rhombus.

20. AB, AC are two tangents to a ⊙, and make with the chord BC an equilateral △: prove that the line joining A to the centre of ⊙ is bisected by the circumference.

21. A number of triangles of equal vertical angles are inscribed in the same circle: show that their bases are all tangents to a fixed circle.

Miscellaneous Exercises.—III.

1. If A and B be two given points on the same side of a given straight line CD, show how to find a point P in CD, or CD produced, such that AP + PB is the least possible.

2. In the base BC of △ ABC any point D is taken: draw a straight line such that if △ ABC be folded along this line, the point A comes upon the point D.

3. On a given straight line construct a △ having one of its ∠s adjacent to this side equal to a given ∠, and having the sum of its other two sides equal to another given straight line.

4. The diagonals AC, BD of a quadrilateral figure ABCD intersect at O: if △ ABC = 2 △ ADC, prove that OB = 2 OD.

5. Show that, if two pairs of opposite ∠s of a hexagon inscribed in a circle be equal, the third pair of opposite ∠s will also be equal, and opposite sides will be parallel.

6. The sides AB, AD of the ▱ ABCD are bisected at E and F respectively: prove that the straight line EF is parallel to the diagonal BD, and that the straight lines CE, CF trisect BD.

7. AOB, COD are two straight lines: find a point P in AB or AB produced such that CP, DP make equal ∠s with AP.

8. Describe a circle to pass through two given points and touch a given straight line. When is this impossible?

9. **ABCDEFGH** is a regular octagon, and **AF, BE, CH, DG** are drawn : prove that their points of intersection are the corners of a square. [NOTE.—A regular octagon has eight equal sides and eight equal angles.]

10. **ABC** is a triangle ; the base **BC** is bisected at **P**, **AP** is bisected at **Q**, and **BQ** is joined and produced to meet **AC** in **R** : prove that **QR** is $\frac{1}{3}$ **BQ**.

11. **ABC** and **DBC** are two right-angled $\triangle$s on opposite sides of a common hypotenuse **BC** ; **AD** is joined, cutting **BC** in **E**, and **EP, EQ** are drawn $\perp$ to **AB** and **BD** : prove that **PQ** and **AD** are $\parallel$.

12. From the angular points **B, C** of a $\triangle$ **ABC** perpendiculars **BQ, CR** are drawn to **AC, AB**, intersecting in **P** : prove that **QR** is $\perp$ to the straight line joining the middle points of **AP** and **BC**.

13. Four points are taken, one on each side of a square, so that the lines joining the points on opposite sides are perpendicular to one another : show that these lines are equal.

14. **ABC** is an equilateral $\triangle$ inscribed in a $\odot$; **E, F** are points on **AB, AC** produced, such that **EB, FC** are each equal to **AB** : show that the four points **B, C, E, F** lie on a circle which cuts the former circle at right angles.

15. If **O** be the centre of the inscribed circle of a $\triangle$, and O_1, O_2, O_3 the centres of the escribed circles, prove that **O** is the orthocentre of the $\triangle$ $O_1O_2O_3$.

NOTE.—The **orthocentre** of a $\triangle$ is the point of intersection of $\perp$s from the vertices to the opposite sides.

16. **BAC** is a right-angled $\triangle$, **A** being the right angle ; **ACDE, BCFG** are the squares on **AC** and **BC** ; **AC** produced meets **DF** in **K** : prove that **DF** is bisected in **K**, and that **AB** is double **CK**.

17. Prove that if the alternate sides of a regular pentagon are produced to meet, the angles of the star-shaped figure thus formed are each two-fifths of a right angle.

18. In a regular heptagon the extremities of a side **DE** are joined to the opposite vertex **A** : prove that in the triangle **ADE**, each angle at the base is three times the vertical angle.

19. In a triangle **ABC** the perpendiculars **AD, BE**, drawn from two vertices to the opposite sides, meet in a point **O**, and **AD** meets the circle circumscribed to the triangle in a point **K** : prove that **DK** is equal to **DO**.

20. Construct $\triangle$ with $\angle$s equal to those of a given $\triangle$, and area four times the area of the given $\triangle$.

21. Upon the same base AB and upon opposite sides of it the parallelograms ABCD and ABEF are described so that the side AD of the first is equal to the diagonal AE of the second, and the diagonal AC of the first is equal to the side AF of the second: prove that AC and AF, and also AD and AE, are in the same straight line.

22. Make a triangle equal in area to the difference between any two given triangles.

23. O is any point within a square ABCD; on OA a square AOEF is described, on the same side of it as D: prove that BO = DF.

24. Having given the base of a triangle in position and magnitude, and also the magnitude of the vertical angle, prove that the centre of the circumscribed circle of the triangle is at a fixed point, and that the centre of each escribed circle lies on a fixed circle.

25. Given the base and orthocentre of a $\triangle$, construct the $\triangle$.

26. P is a point within the square ABCD such that the angles APB, CPD are right angles: prove that P must be at the centre of the square.

ANSWERS TO EXERCISES

Section I. Part II.

Page 126.

3. 9·8″, 9·17″, 8·66″, 6·55″. 4. 3 ft. 2 in.

Page 144.

2. 4″. 3. 17·03 cms. 4. 12·53 cms. 5. 3·21″.
6. 483 ft., 810 ft. 7. 179 miles from balloon to horizon.

END OF SECTION I. PART II.

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